Contribution à la modélisation de phénomènes de frontière libre en mécanique des films minces

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N^o d'ordre 2005-ISAL-0082

Thèse

CONTRIBUTION À LA MODÉLISATION DE PHÉNOMÈNES DE FRONTIÈRE LIBRE EN MÉCANIQUE DES FILMS MINCES

présentée et soutenue publiquement par

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le 21 novembre 2005,

devant

l'**Institut National des Sciences Appliquées de Lyon**

pour l'obtention du DIPLÔME DE DOCTORAT.

Ecole doctorale : MathIF (Mathématiques et Informatique Fondamentale)

Spécialité : Mathématiques Appliquées.

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Cette thèse de doctorat a été réalisée au sein du laboratoire :

INSTITUT CAMILLE JORDAN, CNRS-UMR 5208,
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Remerciements

Tout d'abord, je tiens à remercier Guy Bayada et Carlos Vázquez pour l'opportunité d'une telle collaboration, pour la richesse des sujets qu'ils m'ont proposés et les questions qu'ils m'ont soumises. Cette thèse n'aurait pas été possible sans leurs conseils avisés, leur vigilance et la qualité de leur encadrement.

Je remercie vivement Doina Cioranescu et Danielle Hilhorst, qui m'ont fait un grand honneur en acceptant d'être les rapporteurs de cette thèse, ainsi que Didier Bresch et Jesús Ildefonso Díaz pour leur participation à ce jury : leurs travaux ont eu (et ont encore) une influence importante sur ma formation scientifique. Par ailleurs, j'aimerais exprimer ma reconnaissance à Marie-Hélène Meurisse et Ton Lubrecht pour les discussions sur les aspects mécaniques et pour l'intérêt qu'ils ont porté à ce travail. Mohand Moussaoui a été un professeur passionnant à l'Ecole Centrale de Lyon, et m'a donné l'envie de faire de la recherche. Je lui en suis très reconnaissant.

Je voudrais exprimer ma reconnaissance envers Michèle Chambat pour la lecture attentive et les corrections de ce manuscrit, ainsi que Stéphane Balac et Olivier Mazet qui ont relu certaines pages de ce document et qui m'ont aidé à préparer la soutenance de cette thèse. Il est évidemment plaisant de penser à ceux que j'ai côtoyés au quotidien durant ces trois années et avec qui j'ai eu de nombreuses discussions scientifiques et extra-scientifiques aussi bien à Lyon[1] qu'à La Corogne[2]. L'accueil à La Corogne et à Vigo fut des plus chaleureux : en particulier, j'aimerais exprimer ma profonde gratitude à Iñigo Arregui, Luis Hervella et Jose Durany.

Laurent Chupin, Clément Mouhot et Julien Vovelle ont été d'un grand secours et d'une grande disponibilité. J'espère que nos collaborations porteront leurs fruits : elles sont très motivantes.

L'enseignement effectué au cours de ces trois années fut un véritable plaisir. Je remercie évidemment les personnes qui m'ont guidé dans ces activités pédagogiques, en particulier Marie-Pierre Noutary, Aline Aymes et (à nouveau) Stéphane Balac, pour leurs conseils et leur qualité d'écoute.

Je remercie la région Rhône-Alpes pour sa politique scientifique qui a permis de financer partiellement mon séjour de six mois à La Corogne.

Enfin, pour leur soutien sans faille et leur aide, je remercie mes amis et ma famille, en particulier Elodie qui m'a (sup)porté pendant ces années.

[1] en particulier Franck Fontvieille, Roger Gamouana, Imad Hafidi, Youssef Rouchdy, Jalila Sabil, Pascale Stéphan et Frédéric Sturm

[2] en particulier Joaquin Argibay, Javier Farto, Pablo Ferreira-Alemao, Cristina Gonzalez, Dani Mera, Elisa Molanes, Alvaro Muinelo, Daniel Sempere et Francisco José Rodriguez

Préambule

Description de la thèse :

Cette thèse est consacrée à l'analyse mathématique, à la modélisation et au calcul scientifique des problèmes d'interface dans des milieux fluides de faible épaisseur. Les problèmes d'interface liquide-gaz de type cavitation apparaissent dans la plupart des mécanismes lubrifiés et leur modélisation a toujours été un sujet très discuté en tribologie. Celle-ci a initialement utilisé (et utilise encore) des inéquations variationelles mais l'inadéquation de ce modèle qui est non conservatif a conduit à introduire de manière heuristique une modélisation basée sur un système hyperbolique-elliptique. Cependant, cette nouvelle modélisation fait apparaître, elle aussi, ses limitations dès lors que l'on s'intéresse à des conditions de fonctionnement plus réalistes. Parmi ces limitations, on peut citer :

- la possibilité d'utiliser ce modèle en présence de rugosités des surfaces rigides qui composent le mécanisme. Pour cela, on peut mettre en oeuvre des techniques d'homogénéisation appliquées à une équation non-linéaire en pression-saturation,
- la prise en compte de la déformation élastique de surfaces solides due à la pression hydrodynamique du fluide adjacent. Pour cela, il est habituel en élastohydrodynamique (E.H.D.) de modifier les coefficients de l'équation de l'écoulement par l'introduction d'un terme intégral (déformation du type Hertz). La modélisation de la cavitation intervient dans la partie hydrodynamique et, par suite, sur l'ensemble du couplage.
- la possibilité de justifier ce modèle à partir d'une description bifluide rigoureuse de l'écoulement et d'en déduire ainsi une procédure de calcul du frottement associé à l'écoulement mince.

Nous étudions ces différents aspects qui permettent de justifier la pertinence du modèle de cavitation considéré. Le Chapitre 0, introductif, présente le sujet et résume les résultats obtenus. Les chapitres suivants correspondent chacun à un article. En outre, trois annexes complètent cette thèse : les Annexes B et C reprennent partiellement et développent certains apects (numériques essentiellement) en relation avec des motivations liées à des applications réalistes en tribologie.

- *Chapitre 1 :* Nous généralisons les résultats d'existence et d'unicité de solution pour le modèle d'Elrod-Adams (pour certains types de conditions aux limites), puis nous effectuons l'analyse asymptotique du modèle (homogénéisation par convergence double-échelle), motivée par la prise en compte des rugosités des surfaces rigides.

- *Chapitre 2 :* L'analyse asymptotique est étendue au cadre élastohydrodynamique et en présence de piézoviscosité. Pour l'homogénéisation, nous utilisons ici la méthode d'éclatement périodique. Il s'agit de l'homogénéisation d'un problème dans lequel trois types de nonlinéarités interviennent : cavitation, piézoviscosité et déformation élastique des surfaces, ce qui rend le problème non local.

- *Chapitre 3 :* Nous effectuons l'analyse mathématique des équations quasilinéaires du premier ordre sur un domaine borné et avec des données L^∞. Cette étude est motivée par l'analyse d'un problème particulier de cavitation en lubrification hydrodynamique (voir Chapitre 4).

- *Chapitre 4 :* A partir d'une description bifluide de l'écoulement en film mince, une analyse mathématique et numérique du modèle bifluide est mise en oeuvre. Nous comparons ce modèle au modèle d'Elrod-Adams, ce qui permet de justifier partiellement l'approche bifluide dans la description des phénomènes de cavitation.

- *Annexe A :* Nous donnons des résultats d'homogénéisation du problème de la digue, similaire en de nombreux points au problème de lubrification hydrodynamique (modèle d'Elrod-Adams). Cette étude est motivée par la prise en compte de milieux stratifiés, dont la perméabilité oscille.

- *Annexe B :* A partir du formalisme utilisé en mécanique, nous reprenons l'étude asymptotique développée au Chapitre 1. L'intérêt est motivé par la simulation numérique pour des applications en mécanique (rugosités transverses, longitudinales, bidimensionnelles, obliques, cavitation interaspérités).

- *Annexe C :* Comme dans l'Annexe B, nous établissons les équations limites valables en présence de nombreuses rugosités, dans le cadre élastohydrodynamique, avec le formalisme utilisé en mécanique. Numériquement, notre intérêt se porte également sur des applications réalistes et sur l'influence des rugosités en fonction de la piézoviscosité, de la raideur élastique, de la charge imposée.

Liste des travaux rassemblés dans la thèse :

- Chapitre 1 : Note parue aux **Comptes Rendus de l'Académie des Sciences, Série Mathématique** et article paru dans **Asymptotic Analysis**.
- Chapitre 2 : Article paru dans **Mathematical Models and Methods in Applied Sciences (M3AS)**.
- Chapitre 3 : Article soumis pour publication.
- Chapitre 4 : Article soumis pour publication.
- Appendice B : Article de mécanique paru dans **ASME Journal of Tribology**.
- Appendice C : Article de mécanique soumis pour publication.

Résumé

Cette thèse est consacrée à l'analyse mathématique, à la modélisation et au calcul scientifique des problèmes d'interface dans des milieux fluides de faible épaisseur. Les problèmes d'interface liquide-gaz de type cavitation apparaissent dans la plupart des mécanismes lubrifiés et leur modélisation a toujours été un sujet très discuté en tribologie. Celle-ci a initialement utilisé (et utilise encore) des inéquations variationelles mais l'inadéquation de ce modèle qui est non conservatif a conduit à introduire de manière heuristique une modélisation basée sur un système hyperbolique-elliptique. Cependant, dans le cadre de cette nouvelle modélisation, des problèmes ouverts apparaissent, dès lors que l'on s'intéresse à des conditions de fonctionnement plus réalistes. Parmi ceux-ci, on peut citer :

- la possibilité d'utiliser ce modèle en présence de rugosités. Il s'agit, du point de vue mathématique, de l'homogénéisation d'une équation en pression-saturation,
- la prise en compte de la déformation élastique de surfaces solides due à la pression hydrodynamique du fluide adjacent. Pour cela, il est habituel en élastohydrodynamique (E.H.D.) de modifier les coefficients de l'équation de l'écoulement par l'introduction d'un terme intégral (déformation du type Hertz). La modélisation de la cavitation intervient dans la partie hydrodynamique et, par suite, sur l'ensemble du couplage.
- la possibilité de justifier ou non ce modèle à partir d'une description bifluide rigoureuse de l'écoulement et d'en déduire ainsi une procédure de calcul du frottement associé à l'écoulement mince.

Nous étudions ces différents aspects qui permettent de justifier la pertinence du modèle de cavitation considéré.

Mots-clés : cavitation, modèle (élasto)hydrodynamique d'Elrod-Adams, homogénéisation multi-échelles, éclatement périodique, équations quasilinéaires du premier ordre sur un domaine borné avec données L^∞, modèle bifluide, équation de Buckley-Leverett.

Abstract

This work is devoted to the mathematical analysis, the modelling and the numerical analysis of interfaces problems in thin films mechanics. Liquid-gas interfaces problems such as cavitation appear in most of lubricated devices and the modelling of such phenomena is highly discussed in tribology. Variational inequalities have been widely used (it is still the case) but the non-conservative properties of this model have led to the introduction of an heuristic model based on an elliptic-hyperbolic system (the so-called Elrod-Adams model). However, this modelling also contains some open questions as one focuses on realistic regimes. Among theses difficulties:

- the possibility to use this model when the surfaces are rough. From a mathematical point of view, this deals with the homogenization study of a pressure-saturation problem,
- the elastic deformation of the solid surfaces due to high peaks of the hydrodynamic pressure. For this, the introduction of modified coefficients is widely used in elastohydrodynamic lubrication (E.H.L.): indeed, a nonlocal integral term (Hertz-type deformation) is considered. Cavitation is still taken into account in the hydrodynamic part and, consequently, on the whole coupling.
- the possibility to justify the modelling from a rigorous bifluid description of the flow and get a way to compute friction coefficients in the thin film flow.

We focus on these aspects which allow to conclude on the well-posedness of the cavitation modelling.

KEY-WORDS: cavitation, (elasto)hydrodynamic Elrod-Adams model, multiscale homogenization, periodic unfolding, first order quasilinear equations on bounded domains with L^∞ data, bifluid model, generalized Buckley-Leverett / Reynolds equations.

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Introduction

“ La lubrification est un élément essentiel des sciences technologiques et des applications mécaniques. Elle joue un rôle important partout où des surfaces sont en mouvement relatif les unes par rapport aux autres. Tous les systèmes mécaniques comportent, plus ou moins, des éléments lubrifiés. On peut dire, sans exagération, que bien peu de sujets ont une incidence aussi importante sur les travaux des ingénieurs... Ceci implique des recherches plus poussées dans le domaine de la lubrification elle-même, une formation plus répandue et plus approfondie en matière de lubrification... et une prise de conscience plus générale du potentiel important que présente ce problème, dans tous les domaines de l'industrie. ”

Société Française des Pétroles
La théorie de la lubrification et ses applications. 1974

0.1 Sur la lubrification hydrodynamique et les modèles de cavitation

0.1.1 La lubrification hydrodynamique : l'hypothèse du film mince

La tribologie, science du frottement et de l'usure des matériaux, vise à comprendre et tenter de maîtriser les phénomènes de dégradation des matériaux, essentiellement néfastes dans de nombreux dispositifs industriels. Elle a ainsi pris une importance grandissante depuis la révolution industrielle. La lubrification désigne le contrôle de l'usure des matériaux par l'introduction d'un film fluide qui réduit le frottement entre les surfaces en quasi-contact et en mouvement relatif. Plus particulièrement, la lubrification hydrodynamique concerne les mécanismes pour lesquels la forme et la vitesse relative de deux surfaces en regard engendrent la formation d'un film mince lubrifié continu sous une pression suffisamment élevée pour empêcher le contact. Le point de départ de la théorie de la lubrification hydrodynamique est un article de Reynolds [Rey86], publié en 1886 dans la revue " Philosophical Transactions of the Royal Society ", intitulé *On the theory of lubrication and its application to Mr Beauchamp tower's experiments, including an experimental determination of the viscosity of olive oil.* Dans cet article, Reynolds obtient de manière heuristique l'équation qui porte son nom et qui constitue le socle des études portant sur les écoulements de faible épaisseur.

La transition des équations de Navier-Stokes vers l'équation de Reynolds, proposée dès 1886, a pour but d'utiliser la faible distance qui sépare les deux surfaces d'un contact lubrifié dans les conditions habituelles (quelques dizaines de microns) pour éliminer toutes les variations suivant la direction perpendiculaire aux deux surfaces. En 1904, Sommerfeld [Som04] donne sa première solution à l'équation de Reynolds dans le cas d'un mécanisme convergent-divergent modélisant un palier infiniment long. En 1905, Michell [Mic05] obtient une solution de l'équation de Reynolds prenant en compte des débits de fuite. Cependant, les premières études méthodiques du mécanisme de la lubrification par film mince furent effectuées en 1920 par Hardy and Bircumshaw [HB25] qui employèrent d'abord le terme de " lubrification limite ". D'un point de vue mathématique, des justifications partielles de cette approximation (analogue à la théorie des plaques en mécanique du solide) ont été publiées en 1950 par Wannier [Wan50] et en 1959 par Elrod [Elr60]. Une justification mathématique (par développement asymptotique formel) a été obtenue par Cimatti et Menchi [CM78] en 1978 alors qu'une démonstration rigoureuse a été établie par Bayada et Chambat [BC86] en 1986, et par Nazarov [Naz90] en 1990.

L'idée fondamentale est de considérer un mécanisme lubrifié de géométrie connue (voir, par exemple, FIG.1) et d'en déterminer les performances à partir d'un calcul de la distribution de la pression. L'équation de Reynolds qui régit la pression p dans le mécanisme s'écrit :

$$\nabla \cdot \left(\frac{\rho h^3}{6\mu} \nabla p \right) = \nabla \cdot \left(\rho h (\vec{v_0^+} + \vec{v_0^-}) \right) + \frac{\partial}{\partial t} \left(\frac{\rho h}{2} \right)$$

où ρ est la masse volumique, μ la viscosité, h la hauteur du contact. $\vec{v_0^\pm}$ désignent les vitesses de chacune des faces du contact dans les directions $x = (x_1, x_2)$ et $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$. h est supposée strictement positive (pas de contact entre les surfaces).

Dans un grand nombre de cas, l'équation de Reynolds peut être simplifiée : on considère désormais (sauf mention contraire) le régime établi, stationnaire, d'un fluide de viscosité et de densité constante ; par ailleurs, la vitesse de cisaillement est dirigée dans le sens des x_1 croissants (i.e. $\vec{v_0} = \vec{v_0^+} + \vec{v_0^-} = (v_0, 0)$). L'équation de Reynolds se réduit

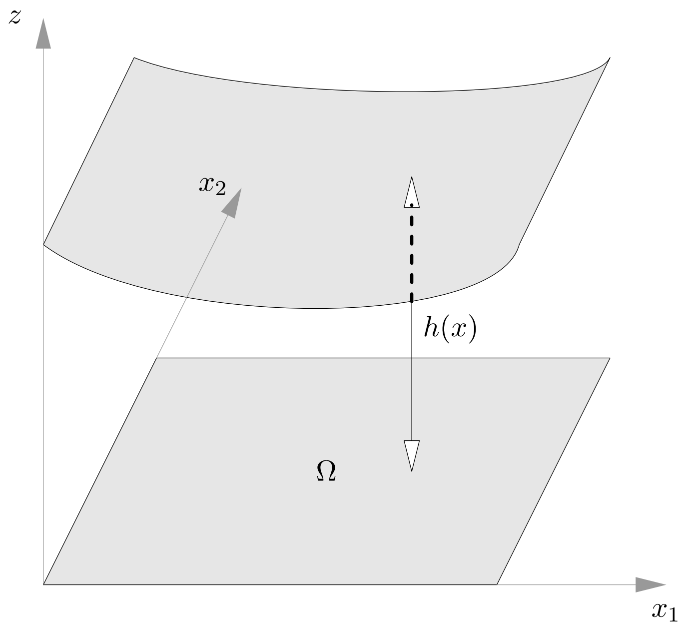


Figure 1. Géométrie d'un contact lubrifié

alors à :

$$\nabla \cdot \left(\frac{h^3}{6\mu} \nabla p \right) = v_0 \frac{\partial h}{\partial x_1}. \tag{0.1}$$

Cette équation doit être complétée par des conditions aux limites adaptées au type de mécanisme étudié : cet aspect sera traité plus loin.

0.1.2 Exemples d'applications et conditions aux limites

On s'intéresse, au moins dans cette introduction, à une géométrie du type " palier lisse " : un palier est un mécanisme qui permet de positionner une pièce mobile par rapport aux autres éléments d'un dispositif. Un palier lisse (voir Fig.2, 3 et 5) est composé de deux cylindres emboîtés coaxiaux mais de section plane non concentrique. Les données géométriques sont les suivantes : les cylindres sont de longueur L et leur section plane est de rayon R_b (resp. R_j) pour le cylindre externe (resp. interne). On définit ainsi le rayon moyen de la section du dispositif $R_m = (R_b + R_j)/2$. Les sections n'étant pas concentriques, cette configuration introduit, lorsque l'on " développe " le mécanisme réel, un espace convergent-divergent entre les deux surfaces en regard. Plus précisément, la hauteur (ou séparation entre les surfaces), dans la configuration développée, est approchée par l'expression :

$$h(x) = c + e\cos(x_1/R_m)$$

où $c = R_b - R_j$ désigne le jeu radial et e est l'excentricité (distance entre les centres des sections planes). Le cylindre externe est fixe, alors que le cylindre interne est animé d'une vitesse de rotation $\omega = v_0/2\pi$.

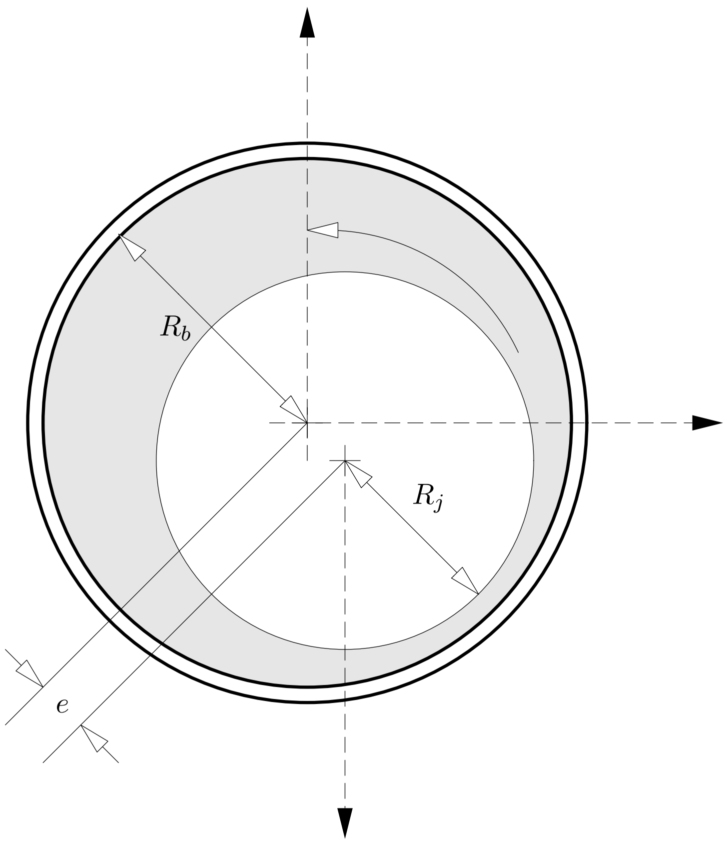


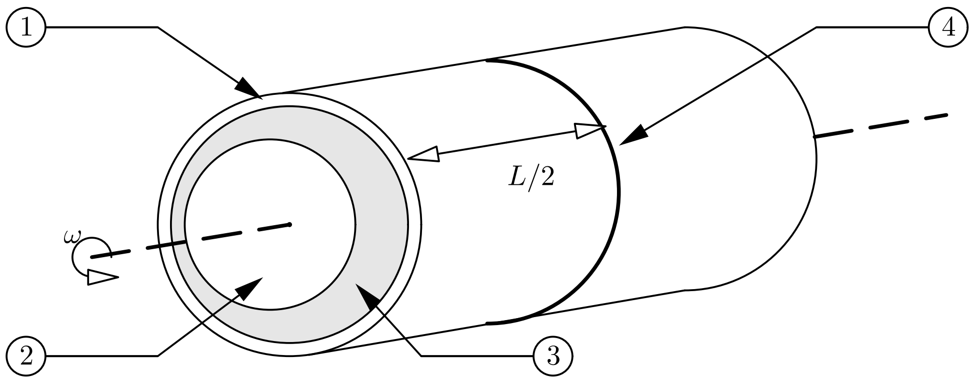
Figure 2. Palier lisse (section)

Le domaine réel bidimensionnel, correspondant à un " développement fictif " du dispositif (voir FIG.4 et 6), est $\Omega =]0, 2\pi R_m[\times]0, L/2[$ ou $\Omega =]0, 2\pi Rm[\times]0, L[$, selon le type d'alimentation.

Il existe essentiellement deux types de conditions aux limites usuels associés à l'équation de Reynolds, qui sont développés ci-après.

Conditions de Dirichlet / périodiques (alimentation circonférentielle)

Une pression p_a (resp. p_0) est imposée en Γ_a (resp. Γ_0) et on impose la périodicité de la pression sur les frontières latérales, ce qui correspond à la continuité du débit et de la pression lors du développement fictif du mécanisme en une coupe axiale arbitraire (voir



1 cylindre externe, de rayon R_b

2 cylindre interne, de rayon R_j

3 lubrifiant

4 rainure d'alimentation circonférentielle

Figure 3. Géométrie réelle d'un palier à alimentation circonférentielle

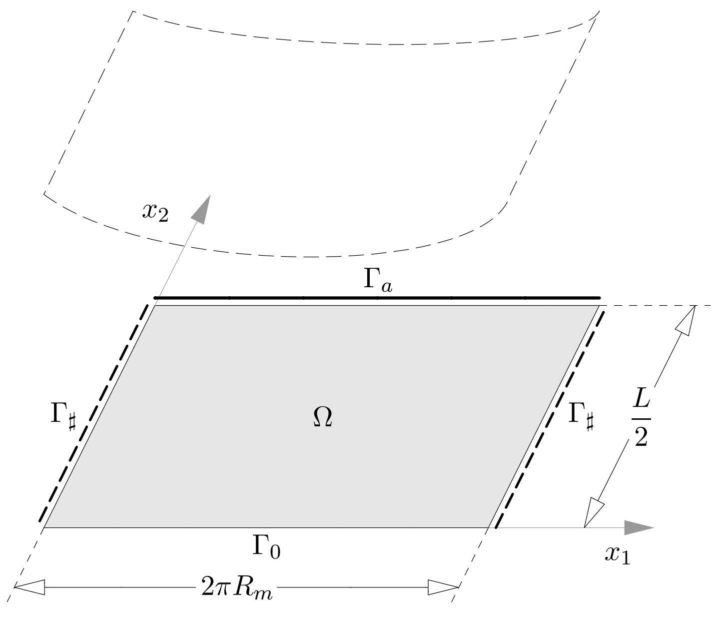
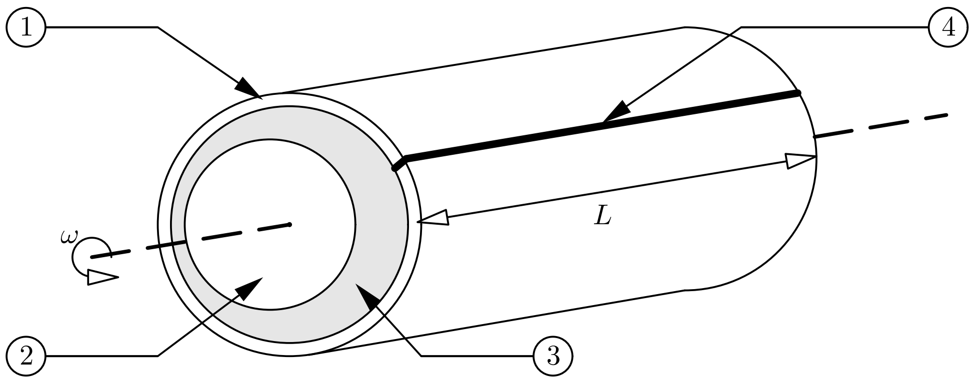


Figure 4. Palier développé (alimentation circonférentielle)

FIG.3 et 4). Les conditions aux limites sont donc :

$$p = p_a \text{ sur } \Gamma_a \text{ et } p = p_0 \text{ sur } \Gamma_0,$$

$$p \text{ et } v_0\, h - \frac{h^3}{6\mu}\frac{\partial p}{\partial x_1} \text{ sont } 2\pi R_m \text{ périodiques dans la direction } x_1.$$



① cylindre externe, de rayon R_b ③ lubrifiant

② cylindre interne, de rayon R_j ④ rainure d'alimentation axiale

Figure 5. Géométrie réelle d'un palier à alimentation axiale

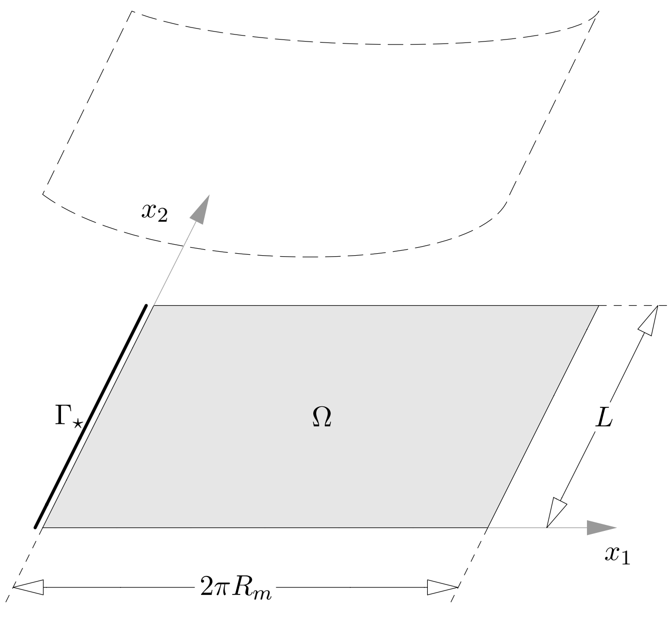


Figure 6. Palier développé (alimentation axiale)

Conditions de Neumann (alimentation axiale)

On impose un flux d'entrée sur $\Gamma_\star$ et une pression (nulle) sur le reste de la frontière, soit :

$$\begin{aligned} &v_0\ h - \frac{h^3}{6\mu}\frac{\partial p}{\partial x_1} = v_0\ \theta_\star h && \text{sur } \Gamma_\star, \\ &p = 0 && \text{sur } \partial\Omega \setminus \Gamma_\star. \end{aligned}$$

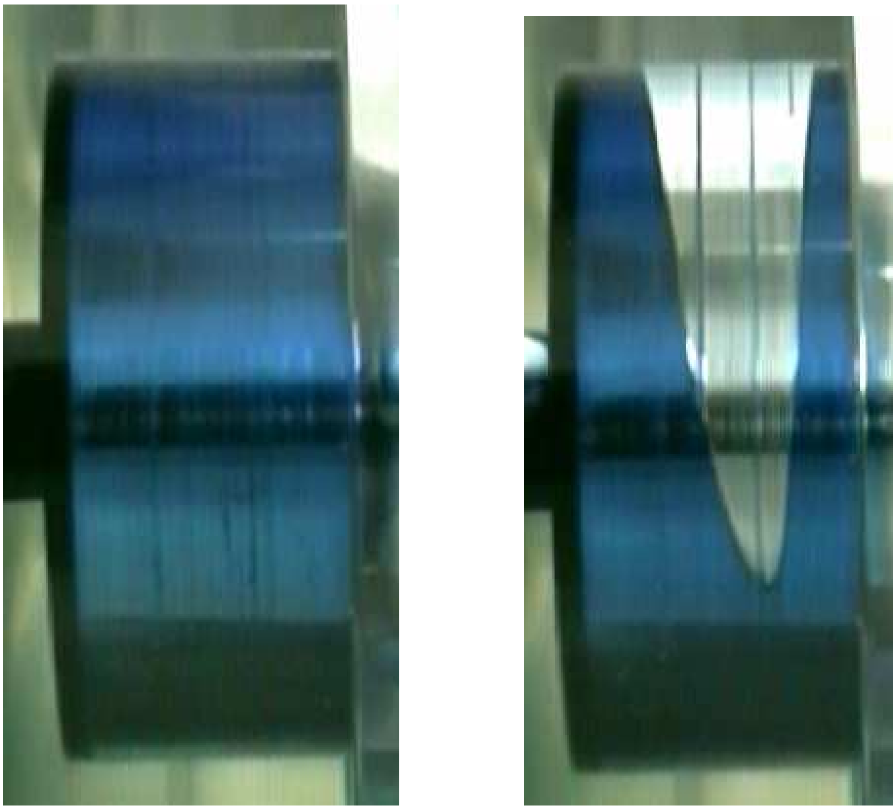


Figure 7. Coussinet en régime lubrifié (g) / partiellement lubrifié (d)

0.1.3 Lubrification partielle : cavitation (rupture du film)

Les photographies de la Fig.7 représentent un mécanisme composé de deux cylindres d'axes parallèles et de sections non concentriques ; les cylindres sont emboîtés et la distance entre la surface interne du cylindre extérieur et la surface externe du cylindre intérieur est faible (préservant toutefois l'absence de contact). Le cylindre extérieur est fixe, tandis que le cylindre intérieur est animé d'une rotation autour de l'axe du cylindre extérieur. Dans l'interstice (supposé de faible épaisseur), un lubrifiant est injecté. Sur la photographie de gauche, on observe une répartition uniforme du film lubrifiant : le film est complet, i.e. l'interstice entre les deux surfaces est complètement rempli de lubrifiant et l'équation de Reynolds est alors considérée comme valide. Sur la photographie de droite, on observe deux zones distinctes : une zone dans laquelle le film est complet et une autre dans laquelle le film n'est que partiel : c'est le phénomène de cavitation, que nous allons développer ci-après.

Les travaux mentionnés dans la section précédente supposent le film complet. En particulier, ceux de Sommerfeld et Michell ne prennent pas en compte une éventuelle rupture du film lubrifié dans le palier : dans la partie divergente, la pression calculée par Sommerfeld peut être inférieure à la pression de saturation p_s, ce qui n'a pas de sens physique. Les changements de phase doivent en effet être pris en compte afin de décrire le fonctionnement d'un mécanisme lubrifié : ainsi, lorsque la pression du fluide atteint la pression de vapeur saturante, des bulles de gaz se forment. Ce phénomène, appelé cavitation (diphasique), modifie considérablement les performances des mécanismes et fait l'objet de nombreuses études mathématiques et physiques. Le domaine d'étude dans ce cas fait intervenir deux zones distinctes :

- une zone non cavitée ou saturée, notée Ω_+, dans laquelle la pression p satisfait $p > p_s$; le film est complet, i.e. l'interstice situé entre les deux surfaces est localement rempli (ou saturé) de lubrifiant liquide, et l'équation de Reynolds est considérée comme valide dans cette zone,
- une zone cavitée, notée Ω_0, dans laquelle la pression p satisfait $p = p_s$; le film n'est pas complet, i.e. l'interstice situé entre les deux surfaces est localement rempli d'un mélange diphasique de lubrifiant (phases liquide et gazeuse), et l'équation de Reynolds n'est pas valide dans cette zone.

Ces deux zones sont séparées par une frontière libre, notée (Σ). L'étude des problèmes à frontière libre relatifs aux écoulements hydrodynamiques en mécanique des films minces a donné lieu à de nombreux travaux couvrant des aspects fondamentaux ou appliqués de la cavitation et de ses effets. Le phénomène physique correspondant est complexe, décrit par un vocabulaire varié (séparation, doigts d'huile, bulles...) lié à la diversité des conditions expérimentales dans lesquelles il apparaît [Dow63, Pan80, PI81]. Par ailleurs, on distingue les “ zones de cavitation de séparation ”, qui sont en contact avec un bord du mécanisme maintenu à pression ambiante, des “ zones de pleine cavitation ” qui sont localisées à l'intérieur du mécanisme. Dans le premier cas, la pression est maintenue à la pression ambiante $p = 0$ tandis que dans le deuxième cas, elle est maintenue à la pression de vapeur saturante $p = p_s < 0$. Néanmoins, dans ce qui suit, cette distinction ne sera pas prise en compte : en effet, les pics de pression observés permettent de justifier cette approximation :

$$\frac{p_s}{\max(p)} \ll 1,$$

autrement dit, $p_s \approx 0$ si l'on se réfère à l'échelle des pics de pression considérés dans les mécanismes étudiés.

En raison de la prépondérance de deux des dimensions caractéristiques du phénomène par rapport à la troisième, on persiste à travailler avec l'équation de Reynolds introduite précédemment, celle-ci étant valide lorsque le film est continu. Cependant, on cherche désormais à intégrer dans cette équation bidimensionnelle le plus grand nombre possible de paramètres physiques afin d'obtenir une solution représentative ou significative de la réalité tridimensionnelle. On obtient ainsi de nombreux modèles décrivant localement la cavitation, mais la nécessité d'obtenir des informations globales sur les performances des mécanismes lubrifiés a conduit à ne retenir parmi ces modèles que ceux dont une mise en oeuvre numérique efficace est aisée.

Afin de décrire les principaux modèles, nous considèrerons ici un palier lisse adimensionné de surface développée $\Omega =]0, 2\pi[\times]0, 1[$, à alimentation axiale située au maximum d'épaisseur du film (voir FIG.1). La pression d'alimentation, la pression atmosphérique et la pression de cavitation seront supposées égales à 0. La hauteur entre les deux surfaces est de la forme $h(x) = 1 + \sigma\cos(x_1)$, $0 < \sigma < 1$. Les conditions aux limites sont de type Dirichlet homogène sur la frontière Γ de Ω :

$$p = 0 \text{ sur } \partial\Omega. \tag{0.2}$$

Le modèle de Sommerfeld

Après l'obtention par Reynolds de l'équation fondamentale qui régit les écoulements de faible épaisseur en simplifiant les équations de Navier-Stokes, Sommerfeld a obtenu une solution analytique pour un palier infiniment long (cas de la dimension 1) avec une excentricité constante et fixe. Le modèle consiste simplement à résoudre les équations (0.1) et (0.2). L'absence de contact entre les deux surfaces et la régularité de h assurent l'existence et l'unicité de la solution. Par ailleurs, une étude détaillée des symétries de la solution a été réalisée par Cimatti [Cim77]. On montre que les pressions négatives sont du même ordre de grandeur que les pressions positives (voir FIG.8), ce qui n'est pas conforme aux observations expérimentales en présence de cavitation.

Le modèle de Gümbel (demi-Sommerfeld)

L'équation de Reynolds est valable dès lors que le film entre les deux surfaces est complet. Afin de réconcilier la solution de cette équation avec les observations expérimentales, Gümbel [G2̈1] avance l'argument selon lequel il est impossible de prendre en compte des pressions inférieures à la pression de vapeur saturante $p_s = 0$. Il a proposé de tronquer la solution de Sommerfeld en relevant les valeurs inférieures à la pression de vapeur saturante à cette valeur p_s (voir FIG.8). Cette proposition est généralement identifiée comme solution de demi-Sommerfeld. Depuis que Gümbel a suggéré cette amélioration, l'intérêt n'a cessé de croître concernant le processus physique de la rupture du film dans les mécanismes. Dans la littérature de la lubrification, le terme de cavitation ou de rupture du film mince est employé afin d'indiquer la présence de la phase gazeuse dans la partie divergente du mécanisme.

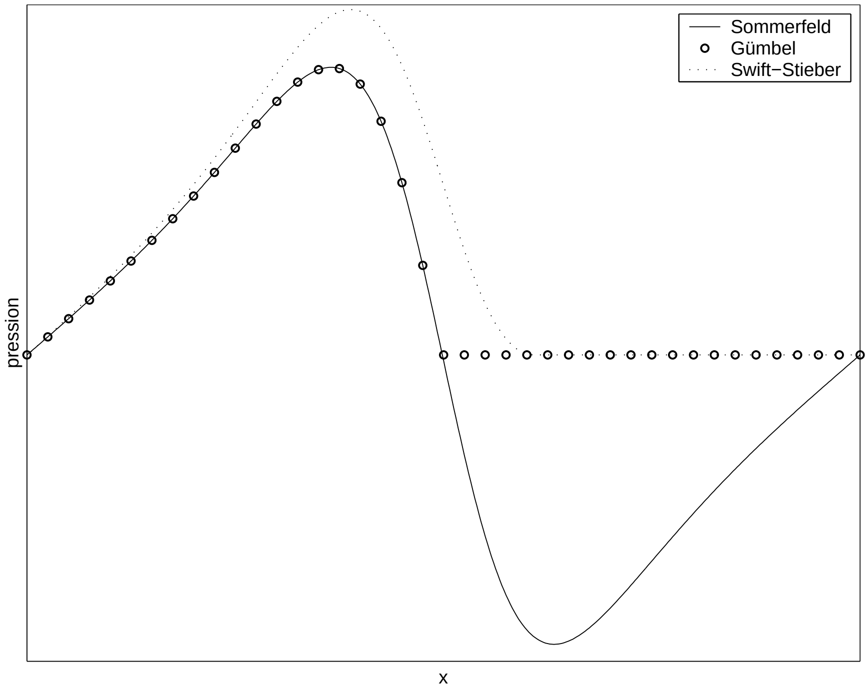


Figure 8. Répartitions de la pression obtenues avec les modèles de Sommerfeld, Gümbel et Swift-Stieber

Le modèle de Swift-Stieber

L'hypothèse de Swift et Stieber [Sti33, Swi32] est que la pression de cavitation est atteinte, au point de rupture, en une zone a priori non localisée où, simultanément, le gradient de la pression s'annule. Ce modèle peut être formalisé comme une *pseudo*-condition de continuité du débit à la rupture du film. En effet, dans la zone active, juste avant la rupture, le débit est donné par :

$$\vec{Q} = \vec{v}_0 \; h - \frac{h^3}{6\mu} \nabla p.$$

Juste après la rupture, il vaut

$$\vec{Q} = \vec{v}_0 \; h. \tag{0.3}$$

D'où la condition :

$$\frac{\partial p}{\partial n} = 0 \tag{0.4}$$

sur la frontière libre. Cette condition permet d'obtenir une répartition de pression beaucoup plus régulière que la condition de Gümbel (voir FIG.8). Elle a par ailleurs été obtenue comme une condition de stabilité par Lamb [Lam93]. Néanmoins, cette condition de continuité est fausse dans la mesure où elle suppose que, dans la zone cavité, le film est complet, ce qui n'est pas le cas ! C'est précisément cet aspect que le modèle d'Elrod-Adams, décrit plus loin, vise à corriger.

En dépit de cette fausse condition de continuité du débit (qui rend le modèle non conservatif), le succès de ce modèle est sans aucun doute lié à l'apparition de la méthode de calcul de Christopherson [Chr41], d'un usage très répandu dans les études de simulation effectuées en lubrification : après discrétisation par différences finies, l'équation (0.1) est résolue par une méthode itérative de type Gauss-Seidel sans tenir compte de la pression a priori ; mais chaque fois qu'une valeur de p négative apparaît, celle-ci est remplacée par la valeur 0. D'un emploi très facile, cette technique numérique a été utilisée de manière intensive afin de calculer des ruptures de films ainsi que des reconstitutions de films, pour des géométries très diverses.

Parallèlement à cette approche, le fait que ce modèle puisse être explicité sous la forme d'une inéquation variationnelle est noté par Lohou [Loh70] et Marzelli [Mar68]. L'application au cas de l'équation de Reynolds des techniques de calcul numérique pour les inéquations variationnelles, développée dans [CM78, LM74], utilise une technique proposée par Stampacchia [Sta72] et basée sur une formulation de type point fixe. Si cette méthode, en termes de performance, est équivalente à celle de Christopherson, l'aspect mathématique des travaux relatifs aux inéquations variationnelles est prépondérant :

- dans le cas du palier à alimentation circonférentielle, Cimatti [Cim77] a établi la forme de la zone de cavitation lorsque la pression d'alimentation est nulle et l'excentricité du palier est faible,

- dans le cas du palier court, Bayada [Bay72] a obtenu une solution analytique au modèle de Reynolds en régime transitoire et démontré la convergence de la solution du problème discret (différences finies variationnelles) pour le palier à alimentation circonférentielle vers la solution du problème continu.

Ces résultats ont validé du point de vue mathématique le modèle physique correspondant, qui a pu être considéré comme un exemple type d'application des inéquations variationnelles [KS00]. Ce modèle a également souvent permis de présenter des méthodes de résolution adaptées à ces problèmes : algorithme d'Uzawa, pénalisation [GLT76], méthode des lignes [Mey81]. Néanmoins, il s'est avéré que l'utilisation extensive de ce modèle pouvait fournir des solutions qui n'étaient pas physiquement satisfaisantes, en raison de la non-conservation du débit dans la formulation. Plus précisément, la validité de l'équation

(0.3) n'est pas établie. Elle signifie simplement que l'on se trouve dans une zone où la pression est constante et ne fait pas intervenir l'existence d'une zone cavitée. Par ailleurs, d'un point de vue numérique, le calcul des quantités de fluide entrant et sortant dans des paliers à rainure circonférentielle et dans les joints [Loh70] montre que ces valeurs peuvent être très différentes. Ainsi, la condition (0.4) comme condition de continuité de débit ne peut être retenue, au moins dans le cas général.

Le modèle de Floberg-Jakobsson-Olsson (JFO)

Une équation de continuité à la rupture du film a été proposée en 1957 par Floberg et Jakobsson [FJ57] puis en 1967 par Olsson [Ols65]. Elle requiert l'introduction d'un modèle diphasique : le modèle introduit en effet une quantité θ, comprise entre 0 et 1, car *" il n'est pas certain que l'huile remplisse toute la largeur du palier dans la zone cavitée "* [FJ57]. Les auteurs proposent alors l'expression suivante pour exprimer le débit dans la zone de cavitation :

$$\vec{Q} = \vec{v}_0 \ \theta h. \tag{0.5}$$

Considérant le cas de la rupture du film, et en supposant que la pression dans la zone cavitée est la plus petite pression admissible, ils en déduisent à l'interface :

$$\frac{\partial p}{\partial n} = 0 \quad \text{et} \quad \theta = 1,$$

n étant la normale extérieure à la zone saturée. Ceci permet de retrouver la condition (0.4). En revanche, en considérant le cas de la reconstitution du film, certaines difficultés apparaissent : dans une région de cavitation qui a la forme d'une bulle, la continuité du débit en tout point de l'interface fournit une relation entre la frontière de rupture et la frontière de reconstitution du film dans laquelle la quantité θ n'apparaît plus, ce qui rend le modèle d'un usage limité et difficile. Les résultats numériques relatifs à ce modèle sont peu nombreux et peu explicites du point de vue de la méthode employée. Par ailleurs, l'aspect purement bidimensionnel et les difficultés d'interprétation de la variable θ ont donné lieu à de nombreuses controverses [DGT75, Flo73].

Le modèle d'Elrod-Adams

L'approche d'Elrod et Adams [Elr81, EA75] reprend le modèle suggéré par Jakobsson, Floberg et Olsson sous une forme qui le rendait difficilement utilisable. Le modèle se distingue du précédent par le rôle prépondérant de la variable θ qui prend implicitement en compte l'aspect tridimensionnel de l'écoulement. En effet, la quantité $\theta(x)$ définit la proportion locale de fluide au point x entre les deux surfaces. Le débit, en un point quelconque du domaine, s'écrit :

$$\vec{Q} = \vec{v}_0 \; \theta h - \frac{h^3}{6\mu} \nabla p.$$

On retrouve alors la condition (0.4) sur l'interface de rupture. A l'interface de formation du film, on obtient

$$p = 0 \quad \text{et} \quad \frac{h^3}{6\mu} \frac{\partial p}{\partial n} = v_0 \; (1 - \theta) h \; \cos(\vec{n}, \vec{x_1}).$$

En particulier, les différentes zones sont caractérisées de la manière suivante :

- dans les zones saturées, $p > 0$ et $\theta = 1$,
- dans les zones cavitées, $p = 0$ et $0 \leq \theta \leq 1$.

Une étude mathématique complète de ce modèle a été faite par Bayada et Chambat pour un palier de longueur finie avec alimentation au maximum d'épaisseur [BC82], pour un palier infiniment long [BC83] et pour un palier à alimentation circonférentielle [BC84]. Les outils mathématiques dans la formalisation de ce modèle sont ceux qui ont été développés pour l'étude de la filtration dans les milieux poreux, en particulier dans le problème de la digue formulé par Brezis, Kinderlehrer et Stampacchia [BKS78], puis Alt [Alt79], en particulier avec la modélisation utilisant une approximation du graphe de Heaviside (méthode de pénalisation) [BKS78] pour l'existence et les méthodes de Carrillo et Chipot [CC81] pour l'unicité. Ces résultats ont été complétés, pour différentes conditions aux limites, par les travaux d'Alvarez et Carrillo [Alv86, AC94] d'une part, Vázquez [Váz94, Váz96] d'autre part. En régime instationnaire, un résultat d'existence et d'unicité a également été établi par Alvarez et Oujja [AO03] pour une alimentation axiale.
Les outils numériques pour la simulation de ce modèle ont été initialement basés sur les travaux initiés par Alt [Alt81], Marini et Pietra [MP86]. Un algorithme a été également développé par Bayada, Chambat et Vázquez [BCV98], utilisant une méthode des caractéristiques [BD87, DGV96a] couplée à une méthode de dualité initiée par Bermúdez et Moreno [BM81].

0.2 Position du problème / description des résultats obtenus

Ce mémoire est consacré à l'analyse mathématique, à la modélisation et à l'analyse numérique de problèmes de lubrification dans lesquels la modélisation de la cavitation intervient par l'introduction de frontières libres décrites par une équation aux dérivées partielles avec discontinuités non linéaires, telles que celles qui ont été décrites dans la Sous-Section 0.1.3. En particulier, nous nous intéressons à différents aspects relatifs au modèle conservatif de cavitation proposé par Elrod. Celui-ci est considéré comme le modèle le plus précis dont disposent actuellement les mécaniciens et permet d'obtenir des résultats satisfaisants par rapport aux expériences.

Les deux premiers chapitres sont consacrés à l'analyse mathématique de problèmes issus du modèle d'Elrod-Adams. En particulier, on s'intéresse aux aspects multi-échelles (homogénéisation) qui interviennent naturellement lorsque les surfaces des mécanismes présentent des rugosités. Ce travail inclut aussi le cas de possibles déformations élastiques de ces surfaces (couplage fluide-structure) qui introduit des termes non locaux.

Les deux autres chapitres correspondent à une tentative de justifier de manière un peu plus physique l'équation proposée dans la Sous-Section 0.1.3 par Elrod. L'idée est de considérer que cette équation décrit un problème à surface libre dans un écoulement de faible épaisseur. Cette surface sépare un fluide (lubrifiant) de viscosité μ_l et un autre de viscosité très faible μ_g assimilé à de l'air. Partant du formalisme des écoulements bifluides développé par Nouri, Poupaud et Demay [NPD97], Bayada, Sabil et Paoli [Sab00, Pao03] ont obtenu, sous certaines hypothèses, par passage à la limite en tenant compte de la faible épaisseur du mécanisme, un système d'équations elliptique-hyperbolique dont nous faisons l'analyse mathématique. Ceci nous a conduit à étudier une équation scalaire avec flux non-autonome sur un domaine borné.

Pour les Chapitres 1, 2 et 4, nous avons développé des algorithmes de résolution permettant de discuter / valider les résultats théoriques obtenus.

Liste des travaux rassemblés dans la thèse :

- Chapitre 1 : Note [BMV05c] parue aux *Comptes Rendus de l'Académie des Sciences, Série Mathématique* et article [BMV05f] paru dans *Asymptotic Analysis.*
- Chapitre 2 : Article [BMV05d] paru dans *Mathematical Models and Methods in Applied Sciences.*
- Chapitre 3 : Article [Mar05] soumis pour publication.
- Chapitre 4 : Article [BMV05a] soumis pour publication.
- Appendice B : Article [BMV05b] paru dans *ASME Journal of Tribology.*
- Appendice C : Article [BMV05e] soumis pour publication.

0.2.1 Résumé du Chapitre 1

Homogénéisation du modèle d'Elrod-Adams (par la méthode de convergence double-échelle)

Dans ce chapitre, nous abordons les aspects suivants:

- Dans un premier temps, nous démontrons l'existence et l'unicité de la solution du problème d'Elrod-Adams pour des conditions aux limites de type alimentation circonférentielle. Cette étude a été réalisée par Alvarez et Carrillo [AC94] dans le cas d'un palier lisse, i.e. lorsque la hauteur h entre les deux surfaces ne dépend que de la variable x_1 (direction du cisaillement). Nous généralisons ici ce résultat pour des configurations géométriques quelconques.
- Dans un deuxième temps, nous nous intéressons à l'influence des rugosités de la surface. La prise en compte de ces défauts de surface nécessite la mise en oeuvre de techniques d'homogénéisation afin d'étudier le comportement du problème en présence de termes fortement oscillants. La difficulté est liée à l'homogénéisation d'un problème non-linéaire de type elliptique-hyperbolique : il s'agit de déterminer les équations limites vérifiées par une pression limite (ce qui est classique) mais aussi par une saturation limite. Cette étude fait apparaître des phénomènes d'anisotropie sur les coefficients (ce qui est classique également), mais aussi sur la saturation, ce qui est nouveau. Des tests numériques permettent d'illustrer ces résultats théoriques.

■ **Section 1** Considérons les hypothèses suivantes, relatives aux données du problème : $h \in C^1(\overline{\Omega})$, bornée, coercive et $2\pi x_1$ périodique, et p_a est une fonction lipschitzienne, non négative, $2\pi x_1$ périodique. La formulation faible du problème est la suivante :

$$(\mathcal{P}_\theta) \begin{cases} \text{Trouver } (p,\theta) \in V_a \times L^\infty(\Omega) \text{ tel que :} \\ \displaystyle\int_\Omega \frac{h^3}{6\mu}\, \nabla p\, \nabla\phi = v_0 \int_\Omega \theta h\, \frac{\partial \phi}{\partial x_1}, \quad \forall\, \phi \in V_0 \\ p \geq 0, \quad p\,(1-\theta) = 0, \quad 0 \leq \theta \leq 1, \quad \text{p.p.,} \end{cases}$$

les espaces fonctionnels V_0 et V_a étant définis par :

$$\begin{aligned} V_0 &= \Big\{\phi \in H^1(\Omega), \quad \phi \text{ est } 2\pi x_1 \text{ périodique}, \quad \phi_{|_{\Gamma_0}} = 0,\ \phi_{|_{\Gamma_a}} = 0\Big\}, \\ V_a &= \Big\{\phi \in H^1(\Omega), \quad \phi \text{ est } 2\pi x_1 \text{ périodique}, \quad \phi_{|_{\Gamma_0}} = 0,\ \phi_{|_{\Gamma_a}} = p_a\Big\}. \end{aligned}$$

Théorème 0.1. *Le problème $(\mathcal{P}_\theta)$ admet au moins une solution (p,θ). De plus, la pression p est unique, et s'il existe un ensemble de mesure positive tel que $p(x_1,x_2) > 0$ pour tout $x_2 > 0$, alors la saturation θ est unique.*

Corollaire 0.2. *Si h s'écrit sous la forme $h(x_1,x_2) = h_1(x_1)h_2(x_2)$, alors le problème $(\mathcal{P}_\theta)$ admet une unique solution.*

Idée de la preuve :

- L'existence d'une solution est établie par une méthode de pénalisation. On considère le problème:

$$(\mathcal{P}_\eta) \begin{cases} \text{Trouver } p_\eta \in V_a \text{ tel que:} \\ \displaystyle\int_\Omega \frac{h^3}{6\mu} \nabla p_\eta \, \nabla \phi = v_0 \int_\Omega H_\eta(p_\eta)\; h \; \frac{\partial \phi}{\partial x_1}, \quad \forall \; \phi \in V_0 \\ p \geq 0, \quad \text{p.p.} \end{cases}$$

  où H_η est une approximation du graphe de Heaviside. L'existence et l'unicité d'une solution pour le problème $(\mathcal{P}_\eta)$ est établie par un point fixe et par le choix d'une fonction test adaptée. Puis, à partir d'estimations en norme H^1, on passe à la limite sur le paramètre de pénalisation.

- L'unicité de la pression est établie grâce à un principe de comparaison obtenu par une méthode de dédoublement de variables, analogue à celle de Kružkov [Kru70], et largement inspirée des résultats obtenus par Alvarez [AC94, AO03]. L'unicité conditionnelle de la saturation se déduit de l'unicité de la pression.

□

■ **Section 2**

L'homogénéisation du modèle d'Elrod-Adams est motivée par la prise en compte des défauts de surface engendrés, volontairement ou non, par les procédés de fabrication. La hauteur entre les deux surfaces en regard est oscillante, les rugosités étant prises en compte par l'introduction d'un petit paramètre ε.

Considérons une hauteur de la forme

$$h_\varepsilon(x) = h\left(x, \frac{x}{\varepsilon}\right),$$

où $h \in L^2(\Omega; C_\sharp(Y))$ (Y désignant la cellule unité $]0,1[\times]0,1[$). Autrement dit, h est une fonction périodique par rapport à sa deuxième variable.

Introduisons explicitement le problème rugueux correspondant :

$$(\mathcal{P}_\theta^\varepsilon) \begin{cases} \textit{Trouver } (p_\varepsilon, \theta_\varepsilon) \in V_a \times L^\infty(\Omega) \textit{ tels que :} \\ \displaystyle\int_\Omega \frac{h_\varepsilon^3}{6\mu} \, \nabla p_\varepsilon \nabla \phi = v_0 \int_\Omega \theta_\varepsilon \; h_\varepsilon \; \frac{\partial \phi}{\partial x_1}, \quad \forall \; \phi \in V_0 \\ p_\varepsilon \geq 0, \quad p_\varepsilon \; (1-\theta_\varepsilon) = 0, \quad 0 \leq \theta_\varepsilon \leq 1, \quad p.p. \end{cases}$$

Le but est de connaître le comportement du problème correspondant $(\mathcal{P}_\theta^\varepsilon)$ et sa (ses) solution(s) lorsque ε tend vers 0, i.e. lorsque le nombre de rugosités tend vers $+\infty$, en utilisant les techniques de convergence double-échelle. Idéalement, on souhaite obtenir un système d'équations limites (ou problème homogénéisé) dont la structure est analogue à celle du problème rugueux. On verra que ce n'est pas toujours le cas.

Après un bref rappel de la notion de convergence double-échelle et de ses principaux outils (voir Définition 1, page 68), on établit les résultats de convergence suivants :

Lemme 0.3. *Il existe $p_0 \in V_a$ tel que, à une sous-suite près :*

$$p_\varepsilon \rightharpoonup p_0 \text{ dans } H^1(\Omega) \qquad \text{et} \qquad p_\varepsilon \to p_0 \text{ dans } L^2(\Omega).$$

Par ailleurs, les convergences suivantes sont établies :

(i) *p_ε converge en double-échelle vers p_0. De plus, il existe $p_1 \in L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$ (ici, $H^1_\sharp(Y)$ désigne l'espace des fonctions dans $H^1(Y)$ périodiques sur la cellule unité $Y =]0,1[\times]0,1[$) et une sous-suite ε', notée ε, tels que ∇p_ε converge en double-échelle vers $\nabla p_0 + \nabla_y p_1$.*

(ii) *Il existe $\theta_0 \in L^2(\Omega \times Y)$ et une sous-suite ε'', notée ε, tels que θ_ε converge en double-échelle vers θ_0.*

(iii) *$p_0 \geq 0$, $0 \leq \theta_0 \leq 1$ et $p_0\ (1 - \theta_0) = 0$ p.p. dans $\Omega \times Y$.*

Idée de la preuve : Les convergences (i) et (ii) sont issues d'estimations indépendantes de ε, tandis que la propriété (iii) reproduit, à l'échelle macroscopique / microscopique, les propriétés de la solution du problème rugueux $(p_\varepsilon, \theta_\varepsilon)$. $\square$

Le but est désormais d'établir un ensemble consistant d'équations vérifiées par ces quantités limites. Dans un premier temps, on établit un résultat partiel, incomplet, que l'on se propose d'approfondir par la suite.

Théorème 0.4. *Le problème homogénéisé s'écrit :*

$$(\mathcal{P}_\theta^\star) \begin{cases} \textit{Trouver } (p_0, \Theta_1, \Theta_2) \in V_a \times L^\infty(\Omega) \times L^\infty(\Omega) \textit{ tels que} \\ \dfrac{1}{6\mu} \displaystyle\int_\Omega \mathcal{A} \cdot \nabla p_0\ \nabla\phi = v_0 \int_\Omega \underline{b}^0 \nabla\phi, \quad \forall\ \phi \in V_0 \\ p_0 \geq 0 \quad \textit{et} \quad p_0\ (1 - \Theta_i) = 0,\ (i = 1, 2) \quad \textit{p.p.} \end{cases}$$

avec $\mathcal{A} = \begin{pmatrix} a_{11}^\star & a_{12}^\star \\ a_{21}^\star & a_{22}^\star \end{pmatrix}$, $\underline{b}^0 = \begin{pmatrix} \Theta_1 b_1^\star \\ \Theta_2 b_2^\star \end{pmatrix}$.

Les coefficients homogénéisés $a_{ij}^\star$ et $b_i^\star$ sont définis par ailleurs de manière explicite *à l'aide de solutions de problèmes locaux. De plus, par construction, le problème homogénéisé admet au moins une solution.*

Idée de la preuve : De manière analogue à la méthode utilisée dans le cadre de l'homogénéisation du problème elliptique modèle [All92, CDG02], on établit une décomposition macroscopique / microscopique donnant la relation entre les fonctions p_0, p_1 et θ_0. Afin de réduire l'équation macroscopique (dans laquelle interviennent les trois fonctions), on résout l'équation microscopique à l'aide de problèmes locaux : cette résolution fournit la relation entre p_1 d'une part, et (p_0, θ_0) d'autre part. Cette relation est alors réinjectée dans l'équation macroscopique, ce qui établit le problème homogénéisé à l'issue d'une normalisation des coefficients (par les coefficients correspondant à $\theta_0(x, \cdot) = 1$, i.e. absence de cavitation). $\square$

Le problème homogénéisé fait apparaître une solution en pression / double-saturation, traduisant les effets d'anisotropie classiques sur les coefficients, mais également sur la saturation. La difficulté essentielle réside dans le fait que nous ne pouvons garantir la propriété $0 \leq \Theta_i \leq 1$ dans les zones cavitées ! Une autre difficulté apparaît lorsque l'on s'intéresse à l'unicité éventuelle d'une solution : les techniques utilisées précédemment ne semblent pas appropriées à ce type de formulation en pression / double-saturation.

Les difficultés sus-mentionnées peuvent néanmoins être partiellement ou complètement dépassées en considérant les résultats suivants :

Théorème 0.5. *Le problème homogénéisé* $(\mathcal{P}_\theta^\star)$ *admet une solution* (p_0, Θ, Θ) *avec, en outre,* $0 \leq \Theta \leq 1$ *p.p. (cette dernière propriété n'est pas établie a priori pour les doubles saturations).*

Idée de la preuve : Le résultat est obtenu en homogénéisant le problème pénalisé rugueux puis en passant à la limite sur le paramètre de pénalisation. $\square$

Autrement dit, le problème homogénéisé admet au moins une solution avec saturation isotrope satisfaisant la propriété $0 \leq \Theta \leq 1$ dans les zones cavitées. A ce stade, en considérant le problème pénalisé, il convient d'observer que l'on ne parvient pas à établir l'équivalence entre les deux démarches suivantes :

- passage à la limite sur le paramètre de pénalisation, puis homogénéisation,
- homogénéisation, puis passage à la limite sur le paramètre de pénalisation.

$$\begin{tikzcd}
(\mathcal{P}_\eta^\varepsilon) \arrow[r, "\eta \to 0"] \arrow[d, "\varepsilon \to 0"'] \arrow[dr, phantom, "(a)\quad (b)"] & (\mathcal{P}_\theta^\varepsilon) \arrow[d, "\varepsilon \to 0"] \\
(\mathcal{P}_\eta^\star) \arrow[r, "\eta \to 0"'] & (\mathcal{P}_\theta^\star)
\end{tikzcd}$$

Figure 9. Homogénéisation des problèmes pénalisé $(\mathcal{P}_\eta^\varepsilon)$ et exact $(\mathcal{P}_\theta^\varepsilon)$

Ainsi, si l'on observe le diagramme de la FIG.9, la démarche (a) établit l'existence d'une solution au problème homogénéisé avec saturation *possiblement* anisotrope et mal définie, tandis que la démarche (b) établit l'existence d'une solution au problème homogénéisé avec une saturation isotrope et bien définie.

En prenant en compte des types de rugosités spécifiques (cas transverse et longitudinal par exemple), on peut toutefois aller plus loin :

Théorème 0.6. *Si $h(x, y)$ peut s'écrire sous la forme $h_1(x, y_1)h_2(x, y_2)$, le problème homogénéisé s'écrit sous la forme :*

$$(\mathcal{P}_\theta^\star)\begin{cases} \text{Trouver } (p_0, \Theta) \in V_a \times L^\infty(\Omega) \text{ tel que} \\ \dfrac{1}{6\mu}\displaystyle\int_\Omega \mathcal{A} \cdot \nabla p_0 \ \nabla\phi = v_0 \int_\Omega \ \Theta \ b_1^\star \ \frac{\partial \phi}{\partial x_1}, \quad \forall \ \phi \in V_0 \\ p_0 \geq 0, \quad p_0 \ (1 - \Theta) = 0, \quad 0 \leq \Theta \leq 1, \quad \text{p.p.} \end{cases}$$

avec les coefficients homogénéisés suivants (et la convention $\widetilde{u}(x) = \displaystyle\int_Y u(x, y) \ dy$) :

$$\mathcal{A}(x) = \begin{pmatrix} a_1^\star(x) & 0 \\ 0 & a_2^\star(x) \end{pmatrix}, \quad a_i^\star(x) = \frac{\widetilde{h_j^3}}{\widetilde{h_i^{-3}}}(x), \quad \text{et} \quad b_1^\star(x) = \frac{\widetilde{h_1^{-2}}\widetilde{h_2}}{\widetilde{h_1^{-3}}}(x).$$

De plus, $(\mathcal{P}_\theta^\star)$ admet (p_0, Θ) comme solution, où

$$\Theta(x) = \Big[\frac{1}{\widetilde{h_1^{-2}}\widetilde{h_2}} \ \widetilde{\Big(\frac{\theta_0 h_2}{h_1^{-2}}\Big)}\Big](x) \tag{0.6}$$

et (p_0, θ_0) est la limite double-échelle de $(p_\varepsilon, \theta_\varepsilon)$ (solution du problème $(\mathcal{P}_\theta^\varepsilon)$).

Des résultats d'unicité pour ce problème homogénéisé sont obtenus sous des hypothèses comparables à celles utilisées dans le cas du problème initial (voir les résultats de la Section 1). Notons par ailleurs qu'une démarche initiale " naïve " aurait pu consister à tenter de déterminer les équations limites vérifiées par la limite faible de la solution de $(\mathcal{P}_\theta^\varepsilon)$, c'est-à-dire

$$(p_0, \widetilde{\theta_0}).$$

Or, si la limite faible de la pression intervient effectivement dans le problème homogénéisé, ce n'est pas la limite faible de la saturation qui joue le rôle de saturation macroscopique, mais une moyenne de θ_0 pondérée par des effets de rugosités dans chaque direction (voir Equation (0.6)).

Par ailleurs, un cas intermédiaire permet de dépasser partiellement les difficultés sus-mentionnées : le cas de rugosités obliques. L'idée qui motive ce résultat est la suivante : l'hypothèse de séparation de variables microscopiques permet de résoudre les difficultés rencontrées dans le cas général. Ainsi, que se passe-t-il lorsque l'on considère cette hypothèse de séparation de variables dans un système de coordonnées en rotation par rapport au système de coordonnées usuel ?

Théorème 0.7. *Supposons que les rugosités sont décrites par*

$$h_\varepsilon(x) = h_1\Big(x, \frac{X_1^\gamma(x)}{\varepsilon}\Big)\ h_2\Big(x, \frac{X_2^\gamma(x)}{\varepsilon}\Big), \ \textit{avec} \begin{cases} X_1^\gamma(x) = & \cos\gamma\ x_1 + \sin\gamma\ x_2 \\ X_2^\gamma(x) = & -\sin\gamma\ x_1 + \cos\gamma\ x_2 \end{cases}$$

Le problème homogénéisé s'écrit alors sous la forme :

$$\begin{cases} \text{Trouver } (p_0, \Theta_1, \Theta_2) \in V_a \times (L^\infty(\Omega))^2 \text{ tel que} \\ \dfrac{1}{6\mu}\displaystyle\int_\Omega \mathcal{A}\cdot\nabla p_0\ \nabla\phi = v_0 \Big(\int_\Omega \underline{b}_1^0\ \frac{\partial\phi}{\partial x_1} + \int_\Omega \underline{b}_2^0\ \frac{\partial\phi}{\partial x_2}\Big), \quad \forall\ \phi \in V_0 \\ p_0 \geq 0, \quad p_0\ (1-\Theta_i) = 0, \quad 0 \leq \Theta_i \leq 1, \quad (i=1,2) \quad \textit{p.p.} \end{cases}$$

avec

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} a_1^\star & 0 \\ 0 & a_2^\star \end{pmatrix} + (a_1^\star - a_2^\star)\ \sin\gamma \begin{pmatrix} -\sin\gamma & \cos\gamma \\ \cos\gamma & \sin\gamma \end{pmatrix}, \\ \underline{b}_1^0 &= b_1^\star\ \Theta_1 - \Big(b_1^\star\ \Theta_1 - b_2^\star\ \Theta_2\Big)\ \sin^2\gamma, \\ \underline{b}_2^0 &= \Big(b_1^\star\ \Theta_1 - b_2^\star\ \Theta_2\Big)\ \sin\gamma\ \cos\gamma. \end{aligned}$$

De plus, $(\mathcal{P}_\theta^\star)$ *admet* $(p_0, \Theta_1, \Theta_2)$ *comme solution, où*

$$\Theta_i = \frac{\theta_0 \widetilde{h_i^{-2} h_j}}{\widetilde{h_i^{-2} h_j}} \quad i, j = 1, 2, \quad j \neq i$$

et (p_0, θ_0) *est la limite double-échelle de* $(p_\varepsilon, \theta_\varepsilon)$ *(solution du problème* $(\mathcal{P}_\theta^\varepsilon)$*).*

Idée de la preuve : En notant $\breve{f}(X) = f(x)$, on a

$$\breve{h}_\varepsilon(X) = \breve{h}_1\Big(X, \frac{X_1}{\varepsilon}\Big)\, \breve{h}_2\Big(X, \frac{X_2}{\varepsilon}\Big).$$

La preuve se construit alors en trois étapes :

▷ *Etape 1* : changement de coordonnées (ré-écriture du problème rugueux),

▷ *Etape 2* : homogénéisation dans les nouvelles coordonnées,

▷ *Etape 3* : retour aux coordonnées initiales. □

On montre ainsi que le problème homogénéisé fait apparaître, comme dans le cas des rugosités obliques, deux fonctions de saturation distinctes. Mais on peut montrer que ces fonctions sont comprises entre 0 et 1 dans les zones cavitées, ce qui n'était pas garanti dans le cas général.

■ **Section 3**

A partir d'un code de calculs développé par Bermúdez, Durany et Vázquez [BM81, BD89, BCV98], l'implémentation et l'adaptation à la prise en compte des effets d'anisotropie ont permis d'illustrer numériquement les résulats de convergence de la pression et le comportement oscillant de la saturation. Les simulations numériques correspondent à des jeux de données réalistes du point de vue des applications en lubrification hydrodynamique.

0.2.2 Résumé du Chapitre 2

Homogénéisation d'un problème non local en lubrification EHL (par la méthode d'éclatement périodique)

Dans ce chapitre, nous étudions l'influence des rugosités de surface sur le comportement de mécanismes tels que des roulements à billes. La différence essentielle concerne les hypothèses de travail : pour de tels mécanismes, il n'est pas raisonnable de considérer les aspects purement hydrodynamiques, en raison des pics de pression qui engendrent de fortes déformations de la surface. Ainsi, la distance effective entre les deux surfaces varie sous l'influence des effets hydrodynamiques et de la réponse élastique des solides, ce qui est pris en compte par l'introduction d'une contribution de déformation, modélisée par un terme non local (modèle de Hertz [Her82]) :

$$h[p](x) = h_r(x) + \int_\Omega k(x,z)\ p(z)\ dz,$$

h_r étant la contribution rigide initiale. Ici, le choix de ce modèle est spécifique au type de contact considéré : linéaire ou ponctuel. Le noyau k pondère l'influence des pics de pression. Par ailleurs, en présence de pics de pression élevés, il est nécessaire de prendre en compte les propriétés piézovisqueuses du fluide avec, par exemple, une loi de Barus [Bar93] $\mu = \mu_0\ e^{\alpha p}$, μ_0 étant la viscosité du lubrifiant à pression ambiante. L'ensemble des équations à considérer est alors le suivant :

$$\nabla \cdot \left(\frac{h[p]^3}{6\mu_0}\ e^{-\alpha p}\ \nabla p \right) = v_0 \frac{\partial}{\partial x_1} \left(\theta h[p] \right), \quad p \geq 0, \quad 0 \leq \theta 1, \quad p(1-\theta) = 0.$$

Pour ce type d'équations, un résultat d'existence mathématique a été établi dans [BEATV96] et [DGV96a] selon les conditions aux limites. L'influence des rugosités pour ce type de problème, par une analyse asymptotique, est un enjeu d'autant plus important que les calculs relatifs à la simulation de ces mécanismes sont coûteux, nécessitant par exemple la mise en oeuvre de méthodes multigrilles [Ven92, VL93, VL00, VtNB90]. La prise en compte des rugosités de surface implique une discrétisation encore plus fine, que l'on peut éviter par l'étude du modèle asymptotique correspondant.

- Dans un premier temps, nous introduisons le problème élastohydrodynamique rugueux et établissons des estimations indépendantes de ε (paramètre modélisant la rugosité), préalable à une analyse asymptotique rigoureuse.
- Dans un deuxième temps, nous établissons une décomposition à l'échelle macroscopique / microscopique du problème, de manière analogue au problème hydrodynamique, en utilisant la méthode d'éclatement périodique [CDG02] : en particulier, dans les termes non-linéaires supplémentaires (piézoviscosité et déformation élastique), seule la pression d'ordre 0 intervient (aussi bien dans l'équation macroscopique que dans l'équation microscopique). Formellement, la résolution se fait de manière analogue au problème hydrodynamique, mais les problèmes locaux (plus précisément, leurs coefficients) dépendent désormais de la pression macroscopique. Nous établissons également l'existence d'une solution avec saturation isotrope : formellement, le résultat est analogue à celui qui a été établi dans le cas hydrodynamique, mais le niveau de difficulté est très différent dans la démonstration du résultat : dans le cas hydrodynamique, les coefficients homogénéisés ne dépendent que de problèmes locaux indépendants ; dans le cas EHL, ces coefficients dépendent de la pression, si bien que les coefficients homogénéisés du problème pénalisé ne sont plus identiques à ceux du problème exact homogénéisé $(\mathcal{P}_\theta^\star)$. Il faut alors montrer que les coefficients homogénéisés pénalisés convergent en un certain sens vers les coefficients homogénéisés du problème exact. Ce résultat est établi en étudiant le comportement de la pression homogénéisée pénalisée. Enfin, nous montrons que dans le cas de rugosités transverses ou longitudinales, le problème ho-

mogénéisé présente une structure analogue à celle du problème initial. Des tests numériques valident les résultats théoriques.

■ Section 1

Hypothèse 1. *La contribution rigide est décrite par :*

$$h_r^\varepsilon(x) = h_r\left(x, \frac{x}{\varepsilon}\right)$$

avec h_r *est une fonction de* $C^0(\Omega \times Y)$, Y *périodique, supérieure à une constante strictement positive. De plus, la hauteur moyenne entre les deux surfaces vérifie :*

$$(0.7) \qquad \int_Y h_r(x,y)\ dy = \begin{cases} h_0 + \dfrac{x_1^2 + x_2^2}{2R}, & \text{pour des contacts ponctuels} \\ h_0 + \dfrac{x_1^2}{R}, & \text{pour des contacts linéaires} \end{cases}$$

R étant le rayon de la sphère ou de la section du cylindre formant la partie supérieure du contact. La distance effective entre les surfaces est donnée par :

$$(0.8) \qquad h_\varepsilon[p](x) = h_r^\varepsilon(x) + \int_\Omega k(x,z)p(z)\ dz, \quad \forall\ x \in \Omega$$

où k est définie par

$$(0.9) \qquad k(x,z) = \begin{cases} c_0\ \log\left|\dfrac{c_1 - z_1}{x_1 - z_1}\right|, & \text{pour des contacts linéaires} \\ \dfrac{c_0}{\sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2}}, & \text{pour des contacts ponctuels,} \end{cases}$$

avec $c_0 > 0$ *et* $c_1 \geq \max\{\left|x_1\right|,\ x \in \bar{\Omega}\}$.

Le problème s'écrit sous la forme :

$$(\mathcal{P}_\theta^\varepsilon) \begin{cases} \text{Trouver } (p_\varepsilon, \theta_\varepsilon) \in V \times L^\infty(\Omega) \text{ tel que :} \\ \displaystyle\int_\Omega \frac{h_\varepsilon^3[p_\varepsilon]}{6\mu_0}\ e^{-\alpha p_\varepsilon}\ \nabla p_\varepsilon\ \nabla\phi = v_0 \left(\int_\Omega \theta_\varepsilon h_\varepsilon[p_\varepsilon]\ \frac{\partial\phi}{\partial x_1} + \int_{\Gamma_\star} \theta_\star h_\varepsilon[p_\varepsilon]\ \phi \right),\ \forall\ \phi \in V \\ p_\varepsilon \geq 0, \quad p_\varepsilon\ (1 - \theta_\varepsilon) = 0, \quad 0 \leq \theta_\varepsilon \leq 1 \quad \text{p.p.} \end{cases}$$

l'espace fonctionnel V étant défini par $V = \left\{\phi \in H^1(\Omega),\ \phi_{|\Gamma} = 0\right\}$. La donnée au bord, $\theta_\star$ est la donnée au bord et vérifie : $\theta_\star \in L^\infty(\Gamma_\star; [0,1])$. En reprenant les travaux de Durany, García et Vázquez [DGV96a], on etablit un résultat d'existence pour ce problème. En outre, on montre que cette solution satisfait des estimations indépendantes de ε en norme H^1 et L^∞ pour la pression et en norme L^∞ pour la saturation.

■ **Section 2**

A partir des estimations précédentes, comme dans le cas hydrodynamique, on établit l'existence de limites double-échelle pour la pression (p_0), le gradient de la pression ($\nabla p_0 + \nabla_y p_1$) et la saturation ($\theta_0$), ainsi que les propriétés reliant p_0 et θ_0. Néanmoins, dans ce chapitre, nous proposons une preuve différente en utilisant le formalisme et les techniques d'éclatement périodique initiées par Cioranescu, Damlamian et Griso [CDG02]. Par suite, on établit :

Théorème 0.8. *Le problème homogénéisé s'écrit :*

$$(\mathcal{P}_\theta^\star)\left\{\begin{array}{l} \textit{Trouver } (p_0,\Theta_1,\Theta_2)\in V\times L^\infty(\Omega)\times L^\infty(\Omega) \textit{ tel que :}\\ \dfrac{1}{6\mu_0}\displaystyle\int_\Omega e^{-\alpha p_0}\ \mathcal{A}[p_0]\cdot\nabla p_0\ \nabla\phi = v_0\left(\int_\Omega \underline{b}^0[p_0]\ \nabla\phi + \int_{\Gamma_\star}\theta_\star\widehat{h[p_0]}\ \phi\right),\ \forall\phi\in V\\ p_0\geq 0 \quad \textit{et} \quad p_0\ (1-\Theta_i)=0,\ (i=1,2) \quad \textit{p.p.}\end{array}\right.$$

avec $\mathcal{A}[p_0] = \begin{pmatrix} a^\star_{11}[p_0] & a^\star_{12}[p_0] \\ a^\star_{21}[p_0] & a^\star_{22}[p_0]\end{pmatrix}$, $\underline{b}^0[p_0] = \begin{pmatrix} \Theta_1\ b^\star_1[p_0] \\ \Theta_2\ b^\star_2[p_0]\end{pmatrix}$.

Les coefficients homogénéisés $a^\star_{ij}[p_0]$ *et* $b^\star_i[p_0]$ *sont définis à l'aide de solutions de problèmes locaux (qui dépendent de* p_0*). De plus, par construction, le problème homogénéisé admet au moins une solution.*

Idée de la preuve : Formellement, on procède de la même manière que dans le cas hydrodynamique. Néanmoins, les nonlinéarités supplémentaires induisent des difficultés que l'on surmonte de la manière suivante :

- En raison de la présence de termes supplémentaires nonlinéaires et non locaux, la décomposition macroscopique / microscopique n'est pas aussi aisée à déterminer que dans le cas hydrodynamique. Cependant, on montre rigoureusement qu'elle est formellement similaire à celle correspondant au cas hydrodynamique, à quelques modifications près : les contributions de la piézoviscosité et de la déformation élastique se réduisent à l'ordre principal en pression (i.e. seule p_0 intervient dans les termes piézovisqueux et élastiques). Ceci est dû en particulier aux propriétés régularisantes du noyau de Hertz ainsi qu'aux estimations L^p $(1\leq p\leq+\infty)$ sur la pression.
- La ré-écriture du problème homogénéisé est obtenue par l'introduction de problèmes locaux, comme dans le cas hydrodynamique. Mais les problèmes locaux dépendent désormais de la pression macroscopique p_0 (de même que les coefficients homogénéisés). □

Comme dans le cas hydrodynamique, nous ne savons pas démontrer que les fonctions de saturation sont à valeurs dans $[0,1]$. Mais, de manière analogue, on montre le résultat suivant :

Théorème 0.9. *Le problème homogénéisé* $(\mathcal{P}_\theta^\star)$ *admet une solution* (p_0, Θ, Θ) *avec, en outre,* $0 \leq \Theta \leq 1$ *p.p. (cette dernière propriété n'étant pas garantie a priori pour les doubles saturations).*

Idée de la preuve : Le résultat est obtenu en homogénéisant le problème pénalisé rugueux puis en passant à la limite sur le paramètre de pénalisation. Techniquement, une difficulté supplémentaire apparaît : le problème pénalisé homogénéisé s'écrit de la manière suivante :

$$\begin{cases} \textit{Trouver } p_0^\eta \in V \textit{ tel que :} \\ \dfrac{1}{6\mu_0} \displaystyle\int_\Omega e^{-\alpha p_0^\eta} \mathcal{A}^\eta[p_0^\eta] \cdot \nabla p_0^\eta \nabla \phi = v_0 \left(\int_\Omega H_\eta(p_0^\eta) \underline{b}^{\eta,\star}[p_0^\eta] \nabla \phi + \int_{\Gamma_\star} \theta_\star \widehat{h[p_0^\eta]} \phi \right), \ \forall \phi \in V \\ p_0^\eta \geq 0 \quad p.p. \end{cases}$$

Remarquons que, contrairement à l'analyse du cas hydrodynamique, les coefficients homogénéisés ne sont pas les mêmes pour les problèmes pénalisé et exact :

- $e^{-\alpha\ \cdot}\ \mathcal{A}^\eta[\cdot]$ et $\underline{b}^{\eta,\star}[\cdot]$ pour le problème pénalisé,
- $e^{-\alpha\ \cdot}\ \mathcal{A}[\cdot]$ et $\underline{b}^{\star}[\cdot]$ pour le problème exact.

Par ailleurs, p_0^η étant bornée en norme H^1 indépendamment de η, converge faiblement vers une limite p_0 dans H^1 (à une sous-suite près). On montre alors le résultat suivant :

$$\begin{array}{ccll} e^{-\alpha\ p_0^\eta}\ \mathcal{A}^\eta[p_0^\eta] & \longrightarrow & e^{-\alpha\ p_0}\ \mathcal{A}[p_0] & \text{dans } L^2(\Omega) \\ \underline{b}^{\eta,\star}[p_0^\eta] & \longrightarrow & \underline{b}^\star[p_0] & \text{dans } L^2(\Omega) \end{array}$$

ce qui permet de conclure la preuve. □

Théorème 0.10 (Rugosités transverses)**.** *Si* $h_r(x,\cdot)$ *ne dépend pas de* y_2*, le problème homogénéisé* $(\mathcal{P}_\theta^\star)$ *s'écrit*

$$\begin{cases} \text{Trouver } (p_0,\ \Theta) \in V \times L^\infty(\Omega) \text{ tel que :} \\ \dfrac{1}{6\mu_0} \displaystyle\int_\Omega e^{-\alpha p_0}\ \mathcal{A}[p_0] \cdot \nabla p_0\ \nabla \phi = v_0 \left(\int_\Omega \Theta\ b_1^\star[p_0]\ \frac{\partial \phi}{\partial x_1} + \int_{\Gamma_\star} \theta_\star \widehat{h[p_0]}\ \phi \right), \quad \forall\ \phi \in V \\ p_0 \geq 0, \quad p_0\ \big(1-\Theta\big) = 0, \quad 0 \leq \Theta \leq 1, \quad \text{p.p.} \end{cases}$$

avec les coefficients homogénéisés :

$$\mathcal{A}[p_0](x) = \begin{pmatrix} \dfrac{1}{\widetilde{h^{-3}[p_0]}(x)} & 0 \\ 0 & \widetilde{h^3[p_0]}(x) \end{pmatrix}, \quad b_1^\star[p_0](x) = \frac{\widetilde{h^{-2}[p_0]}(x)}{\widetilde{h^{-3}[p_0]}(x)}.$$

Ici, on utilise la notation

$$h[p](x,y) = h_r(x,y) + \int_\Omega k(x,z)\ p(z)\ dz.$$

De plus, $(\mathcal{P}_\theta^\star)$ admet au moins une solution (p_0,Θ) où (p_0,θ_0) est la limite double-échelle de $(p_\varepsilon,\theta_\varepsilon)$ (solution de $(\mathcal{P}_\theta^\varepsilon)$), et le lien entre la saturation macroscopique / microscopique est donné par la formule :

$$\Theta(x) = \Big[\frac{1}{\widetilde{h^{-2}[p_0]}}\ \Big(\widetilde{\frac{\theta_0}{h^2[p_0]}}\Big)\Big](x).$$

Théorème 0.11 (Rugosités longitudinales)**.** *Si $h_r(x,\cdot)$ ne dépend pas de y_1, le problème homogénéisé $(\mathcal{P}_\theta^\star)$ s'écrit sous la même forme que précédemment avec les coefficients homogénéisés :*

$$\mathcal{A}[p_0](x) = \begin{pmatrix} \widetilde{h^3[p_0]}(x) & 0 \\ 0 & \dfrac{1}{\widetilde{h^{-3}[p_0]}(x)} \end{pmatrix}, \quad b_1^\star[p_0](x) = \widetilde{h[p_0]}(x).$$

Le lien entre la saturation macroscopique / microscopique est donné par la formule :

$$\Theta(x) = \frac{\widetilde{(\theta_0\ h[p_0])}}{\widetilde{h[p_0]}}(x).$$

- **Section 3** Le code de calculs utilisé dans le cas hydrodynamique a été développé par la suite afin de prendre en compte la déformation élastique (par une méthode de point fixe). Par ailleurs, la piézoviscosité du fluide est prise en compte par une transformation de Grubin-Kirchoff, destinée à faire disparaître la non linéarité due au terme $e^{-\alpha p}$. L'ensemble de ces méthodes a été développé par Durany, García et Vázquez [DGV96b, DGV02]. Nous avons donc implémenté les modifications nécessaires à la prise en compte des effets d'homogénéisation, afin d'illustrer numériquement les résultats de convergence en pression, saturation et déformation.

0.2.3 Résumé du Chapitre 3

Equations quasilinéaires du premier ordre sur un domaine borné avec des données L^∞

L'étude des équations limites pour le modèle bifluide qui sera précisé au Chapitre 4 nous a conduit à nous interroger sur les équations quasilinéaires du premier ordre sur un domaine borné avec des données L^∞. Sommairement, cette étude s'inscrit dans la continuité de nombreux travaux, qu'elle complète ou généralise :

(1) Dans le cadre d'un domaine non borné, ce problème a été étudié par Kružkov [Kru70] qui a introduit le concept d'“ entropie-flux d'entropie (de Kružkov) ” et de solution entropique, afin de fournir un cadre d'existence et d'unicité de solution, et de garantir la pertinence physique de la solution ainsi visée.

(2) Dans le cadre d'un domaine borné, Bardos, Le Roux et Nédélec [BLRN79] ont également établi un résultat d'existence et d'unicité, ainsi que la manière d'interpréter la condition aux limites. En effet, des conditions de Dirichlet ne peuvent être imposées en tout point de la frontière [Vov02b] : toutes les zones de la frontière ne sont pas nécessairement actives et on parlera alors de conditions de Dirichlet “ relaxées ” et, en conséquence, de condition BLN. Néanmoins, cette étude nécessite une régularité BV sur les données, afin de garantir la notion de trace sur le bord.

(3) Dans le cadre d'un domaine borné et avec des données L^∞, les difficultés ont été surmontées par Otto [Ott96] qui a introduit le concept d'“ entropie-flux d'entropie de frontière ” afin de garantir l'unicité de la solution. Notons que ces résultats permettent de retrouver ceux de Bardos, Le Roux et Nédélec et l'interprétation de la condition aux limites. Cependant, l'article d'Otto ne concerne que les lois de conservation scalaires sans terme source et pour des flux autonomes.

Ce chapitre généralise le travail d'Otto, en prenant en compte la présence de termes source et le caractère non-autonome du flux. Par ailleurs, nous établissons un résultat de stabilité par rapport aux données, ce qui n'apparaît pas dans le travail de Bardos, Le Roux et Nédélec. Nous utilisons essentiellement le concept de “ semi entropies-flux d'entropie de Kružkov ” qui est équivalent à celui d'“ entropies-flux d'entropie de frontière ” utilisé par Otto, mais qui est plus approprié à l'étude des lois de conservation avec termes source et flux non-autonomes. Le résultat d'existence est établi par une méthode d'approximation parabolique alors que le résultat d'unicité est obtenu par la méthode du dédoublement de variables. Le résultat de stabilité dépend de certaines hypothèses supplémentaires sur le terme source et le flux de l'équation, conjointement aux données.

Soit Ω un domaine borné régulier de $\mathbb{R}^d$, $d \geq 1$. On s'intéresse aux équations suivantes :

$$\frac{\partial u}{\partial t} + \nabla \cdot \Big(f(t,x,u)\Big) + g(t,x,u) = 0, \quad \text{sur } Q_T = (0,T) \times \Omega, \tag{0.10}$$

$$u(0,\cdot) = u^0, \quad \text{sur } \Omega, \tag{0.11}$$

$$\text{“}u = u^D\text{”}, \quad \text{sur } \Sigma_T = (0,T) \times \partial\Omega, \tag{0.12}$$

où le sens des conditions aux limites, de type Dirichlet, doit être pris en un sens faible (par exemple, au sens de Bardos, Le Roux et Nédélec si la régularité de la solution est suffisante).

Hypothèse 2.

(i) *f et g sont deux fonctions définies sur $[0,T] \times \overline{\Omega} \times \mathbb{R}$ telles que*

$$f \in \left(C^2([0,T] \times \overline{\Omega} \times [a,b])\right)^d, \quad g \in C^2([0,T] \times \overline{\Omega} \times [a,b]),$$

(ii) *f, $\nabla \cdot f$ et g sont lipschitziennes par rapport à u, uniformément en (t,x) (les constantes de Lipschitz étant respectivement notées $\mathcal{L}_{[f]}$, $\mathcal{L}_{[\nabla \cdot f]}$, $\mathcal{L}_{[g]}$),*

(iii) $(u^0, u^D) \in L^\infty(\Omega; [a,b]) \times L^\infty(\Sigma_T; [a,b])$,

(iv) $(\nabla \cdot f + g)(\cdot,\cdot,a) \leq 0$ *et* $(\nabla \cdot f + g)(\cdot,\cdot,b) \geq 0$ *uniformément en* (t,x).

- **Section 1**
  Nous établissons la définition d'une solution faible entropique du problème (0.10)–(0.12) :

  Définition 1. *Supposons l'hypothèse 2 vérifiée. Une fonction $u \in L^\infty(Q_T, [a,b])$ est une solution faible entropique du problème (0.10)–(0.12) si elle vérifie :*

$$(\mathcal{P}_{SK}) \left\{ \begin{array}{l} \displaystyle\int_{Q_T} \left\{ (u-k)^\pm \frac{\partial \varphi}{\partial t} + \big(\mathrm{sgn}_\pm(u-k)(f(t,x,u) - f(t,x,k))\big) \; \nabla\varphi \right. \\ \qquad \left. -\mathrm{sgn}_\pm(u-k)\Big(\nabla \cdot f(t,x,k) + g(t,x,u)\Big) \; \varphi \right\} dx \; dt \\ \qquad + \displaystyle\int_\Omega (u^0 - k)^\pm \; \varphi(0,x) \; dx \\ \qquad + \mathcal{L}_{[f]} \displaystyle\int_{\Sigma_T} (u^D - k)^\pm \varphi(t,r) \; d\gamma(r) \; dt \geq 0 \\ \forall \phi \in \mathcal{D}((-\infty,T) \times \mathbb{R}^d), \; \phi \geq 0, \; \forall k \in \mathbb{R}. \end{array} \right.$$

  Les fonctions $u \mapsto (u-\kappa)^\pm$ sont les “ semi entropies de Kružkov ” [Car99, Ser96, *Vov02a]), définies par*

$$(u-\kappa)^+ = \begin{cases} u-\kappa, & si\ u \geq \kappa, \\ 0, & sinon, \end{cases} \qquad et \qquad (u-\kappa)^- = (\kappa - u)^+,$$

et $u \mapsto \mathrm{sgn}_\pm(u)$ *est la dérivée de la fonction* $u \mapsto u^\pm$ *avec la valeur* 0 *en* 0.

Nous montrons par la suite le résultat de stabilité suivant :

Théorème 0.12 (Stabilité)**.** *Soit* $u \in L^\infty(Q_T)$ *une fonction vérifiant* $(\mathcal{P}_{SK})$. *Alors*

$$a \leq u \leq b \quad \text{p.p.}$$

Idée de la preuve : Le résultat est obtenu en utilisant directement les propriétés des “ semi entropies-flux de Kružkov ”, avec $(u-a)^-$ d'une part et $(u-b)^+$ d'autre part. L'hypothèse 2 (iv) joue ici un rôle essentiel dans ce résultat. □

■ **Section 2**

Nous établissons le résultat d'existence par approximation parabolique. Dans un premier temps, nous considérons le problème :

$$\frac{\partial u_\varepsilon}{\partial t} + \nabla \cdot \Big(f(t,x,u_\varepsilon) \Big) + g(t,x,u_\varepsilon) = \varepsilon \Delta u_\varepsilon, \quad \text{sur } Q_T, \tag{0.13}$$

$$u_\varepsilon(0,\cdot) = u_\varepsilon^0, \quad \text{sur } \Omega, \tag{0.14}$$

$$u = u_\varepsilon^D, \quad \text{sur } \Sigma_T, \tag{0.15}$$

avec l'hypothèse suivante :

Hypothèse 3.

(i) u_ε^D *et* u_ε^0 *vérifient des conditions de compatibilité sur* $\overline{\Sigma}_T \cap \overline{Q}_T$,

(ii) u_ε^D *et* u_ε^0 *sont régulières : par exemple,* $u_\varepsilon^D \in C^2(\Sigma_T;[a,b])$, $u_\varepsilon^0 \in C^2(\overline{\Omega};[a,b])$.

On s'intéresse en particulier au comportement de la solution de ce problème lorsque ε tend vers 0. Le lemme suivant contient deux informations essentielles. D'une part, il établit une version entropique du problème parabolique, qui préfigure celle que l'on obtiendra par passage à la limite pour le problème hyperbolique. D'autre part, il établit des estimations BV qui constituent une étape préalable au passage à la limite sur ε :

Lemme 0.13. *Soit u solution de (0.13)–(0.15) correspondant aux données initiales / aux limites (u^D, u^0) satisfaisant l'hypothèse 3. Alors,*

(i) pour tout $\varphi \in \mathcal{D}(]-\infty, T[\times\mathbb{R}^d)$, pour tout $k \in \mathbb{R}$,

$$\begin{aligned}
&\int_{Q_T} \Big\{ (u-k)^{\pm} \frac{\partial \varphi}{\partial t} + \mathrm{sgn}_{\pm}(u-k)\Big(f(t,x,u) - f(t,x,k)\Big)\nabla\varphi \\
&\qquad -\mathrm{sgn}_{\pm}(u-k)\Big(\nabla\cdot f(t,x,k) + g(t,x,u)\Big)\varphi + \varepsilon\ (u-k)^{\pm}\Delta\varphi \Big\}\ \xi_\varepsilon \\
&\qquad + \int_{\Omega} (u^0-k)^{\pm}\varphi(0,\cdot)\ \xi_\varepsilon \\
&\qquad \geq -2\varepsilon \int_{Q_T} (u-k)^{\pm}\nabla\varphi\nabla\xi_\varepsilon - (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon)\int_{\Sigma_T} (u^D-k)^{\pm}\varphi,
\end{aligned}$$

$\mathcal{R} \in \mathbb{R}$ et $\xi_\varepsilon \in C^0(\overline{\Omega})$ étant choisis de manière appropriée,

(ii) pour tout $t \in (0,T)$, on a

$$\int_{\Omega} \Big| u_1(t,\cdot) - u_2(t,\cdot) \Big|\ \xi_\varepsilon \leq \Big\{ \int_{\Omega} \Big| u_1^0 - u_2^0 \Big| \xi_\varepsilon + (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_{\Sigma_T} \Big| u_1^D - u_2^D \Big| \Big\} e^{\mathcal{L}_{[g]}T},$$

(iii) supposons de plus que u^D a une extension régulière sur $\overline{Q}_T$, notée $\overline{u}^D$. Alors il existe une constante λ qui ne dépend que de $\|u^0\|_\Omega$, $\|\overline{u}^D\|_{\Sigma_T}$, T, Ω, f et g, telle que

$$\sup_{t\in(0,T)} \int_{\Omega} \Big\{ \Big| \frac{\partial u}{\partial t}(t,\cdot) \Big| + \Big| \nabla u(t,\cdot) \Big| \Big\} \leq \lambda.$$

Idée de la preuve :

• Le résultat (i) est obtenu par choix de fonctions test appropriées et en introduisant

$$\mathrm{sgn}_{\pm}^{\eta}(z) = \begin{cases} H_\eta(z), & \text{si } z \in \mathbb{R}^{\pm} \\ -H_\eta(-z), & \text{si } z \in \mathbb{R}^{\mp} \end{cases}, \quad I_\eta^{\pm}(z) = \int_0^z \mathrm{sgn}_{\pm}^{\eta}(t)\ dt$$

qui, à l'évidence, imitent le comportement des “ semi entropies-flux de Kružkov ”. L'inégalité s'obtient directement par passage à la limite sur η. Sans entrer dans les détails techniques, l'introduction de la fonction de pondération ξ_ε vise à conserver le terme de bord lorsque ε tend vers 0.

• Le résultat (ii) exprime une stabilité L^1 “ pondérée ” des solutions du problème parabolique par rapport aux données.

• Le résultat (iii) est obtenu en deux (longues) étapes, par le choix de fonctions test appropriées :

$$
\begin{aligned}
\int_\Omega \left|\frac{\partial u}{\partial t}(t,\cdot)\right| \, dx &\leq c_1 \left(1 + \int_0^t \int_\Omega |\nabla u(\tau,x)| \, dx \, d\tau + \int_0^t \int_\Omega \left|\frac{\partial u}{\partial t}(\tau,x)\right| \, dx \, d\tau\right) \\
\int_\Omega \left|\nabla u(t,\cdot)\right| &\leq c_2 \left(1 + \int_0^t \int_\Omega \left|\nabla u\right| \, d\tau \, dx\right)
\end{aligned}
$$

c_1 et c_2 ne dépendant que des données du problème. La conclusion est immédiate par le lemme de Gronwall. □

Théorème 0.14 (Existence)**.** *Supposons l'hypothèse 2 vérifiée. Soit u_ε l'unique solution de (0.13)–(0.15) correspondant aux données initiales / au bord $(u^0_\varepsilon, u^D_\varepsilon)$ vérifiant l'hypothèse 3 et soit*

$$
\begin{aligned}
\lim_{\varepsilon\to 0} u^D_\varepsilon &= u^D \qquad \text{dans } L^1(\Sigma_T), \\
\lim_{\varepsilon\to 0} u^0_\varepsilon &= u^0 \qquad \text{dans } L^1(\Omega),
\end{aligned}
$$

avec $u^D \in L^\infty(\Sigma_T;[a,b])$ et $u^0 \in L^\infty(\Omega;[a,b])$. Alors, la suite $\{u_\varepsilon\}_\varepsilon$ converge vers une fonction $u \in L^\infty(Q_T;[a,b])$ dans $C^0([0,T],L^1(\Omega))$. De plus, u est une solution faible entropique du problème (0.10)–(0.12).

Idée de la preuve : Afin d'établir l'existence d'une solution, on passe à la limite sur ε. Néanmoins, nous ne pouvons utiliser les estimations du lemme 0.13 (iii) pour u_ε car u^D_ε et u^0_ε vérifient des conditions de compatibilité mais n'ont pas nécessairement une extension sur $\overline{Q}_T$ avec une régularité suffisante. Ainsi, nous introduisons, par construction, $u^D_{\varepsilon,h}$ et $u^0_{\varepsilon,h}$ qui vérifient des conditions de compatibilité et ont une extension sur $\overline{Q}_T$ avec une régularité suffisante. De plus, $u^D_{\varepsilon,h}$ et $u^0_{\varepsilon,h}$ sont uniformément " proches " (en un sens à préciser) de u^D_ε et u^0_ε (lorsque $h \to 0$, uniformément par rapport à ε), ce qui implique que $u_{\varepsilon,h}$ est " proche " de u_ε (en un sens à préciser). Puis, nous appliquons le théorème d'Arzelà-Ascoli afin de démontrer que $\{u_\varepsilon\}$ est relativement compacte dans $C^0([0,T];L^1(\Omega))$. Bien sûr, nous devons démontrer que la suite vérifie les hypothèses du théorème (équicontinuité et relative compacité). Pour cela, nous utilisons les propriétés de $u_{\varepsilon,h}$ et le fait que u_ε est " proche " de $u_{\varepsilon,h}$.□

■ **Section 3**

Nous établissons le résultat d'unicité par la méthode de dédoublement de variables. Au préalable, nous établissons le résultat suivant :

Lemme 0.15. *Soit $u \in L^\infty(Q_T)$ vérifiant $(\mathcal{P}_{SK})$; alors, pour tout $\varphi \in \mathcal{D}((0,T) \times \mathbb{R}^d)$ et pour tout $k \in \mathbb{R}$,*

$$
\begin{aligned}
&\int_{Q_T} \Big\{ \big|u-k\big| \, \frac{\partial \varphi}{\partial t} + \operatorname{sgn}(u-k)\Big(f(t,x,u) - f(t,x,k)\Big) \, \nabla\varphi \\
&\qquad\qquad -\operatorname{sgn}(u-k)\,(\nabla \cdot f(t,x,k) + g(t,x,u)) \;\; \varphi \Big\} \, dx\, dt \\
&\qquad\qquad \geq \int_{\Sigma_T} \operatorname{sgn}(k-u^D)\Big(f(t,r,k) - f(t,r,u^D)\Big) \cdot n(r) \;\varphi(t,r)\; d\gamma(r)\; dt \\
&\qquad\qquad\quad -\operatorname{ess}\lim_{\varrho\to 0^+} \int_{\Sigma_T} \Big\{ \operatorname{sgn}(u(t,r-\varrho\, n(r)) - k)\Big(f(t,r,u(t,r-\varrho\, n(r))) \\
&\qquad\qquad\qquad\qquad\qquad\qquad -f(t,r,k)\Big)\Big\} \cdot n(r)\;\varphi(t,r)\; d\gamma(r)\; dt.
\end{aligned}
$$

Idée de la preuve : Le résultat est obtenu par sommation des deux inégalités liées aux “ semi entropies-flux de Kružkov ”, et en utilisant un choix approprié d'“ entropies-flux de frontière ”. □

Lemme 0.16. *Soit $u \in L^\infty(Q_T)$ (resp. $v \in L^\infty(Q_T)$) une solution de $(\mathcal{P}_{SK})$ avec les conditions initiales / au bord $(u^0, u^D) \in L^\infty(\Omega) \times L^\infty(\Sigma_T)$ (resp. $(v^0, v^D) \in L^\infty(\Omega) \times L^\infty(\Sigma_T)$) ; alors,*

$$
\begin{aligned}
-&\int_{Q_T} \Big\{ \big|u-v\big| \, \frac{\partial \beta}{\partial t} \;\; + \;\; \operatorname{sgn}(u-v)\Big(f(t,x,u) - f(t,x,v)\Big)\nabla\beta \\
&\qquad\qquad\qquad - \;\; \operatorname{sgn}(u-v)\,(g(t,x,u) - g(t,x,v))\,\beta \Big\} \; dx\; dt \\
&\leq \int_\Omega \Big|u^0(x) - v^0(x)\Big| \;\beta(0,x)\; dx + \mathcal{L}_{[f]} \int_{\Sigma_T} \Big|u^D(r) - v^D(r)\Big| \;\beta(t,r)\; d\gamma(r)\; dt
\end{aligned}
$$

pour tout $\beta \in \mathcal{D}((-\infty,T) \times \mathbb{R}^d)$.

Idée de la preuve : Le résultat est obtenu par application de la méthode de dédoublement de variables à la formulation donnée dans le lemme précédent. □

Théorème 0.17 (Unicité)**.** *Sous l'hypothèse 2, $(\mathcal{P}_{SK})$ admet une unique solution faible entropique.*

Idée de la preuve : En utilisant le principe de comparaison précédent, avec $u^0 = v^0$ et $u^D = v^D$, le résultat est obtenu par le lemme de Gronwall. □

0.2.4 Résumé du Chapitre 4

Des équations de Stokes multifluides au modèle d'Elrod-Adams

Dans ce chapitre, nous tentons de justifier la pertinence du modèle d'Elrod-Adams, qui est un modèle heuristique, à partir d'une description bifluide de la cavitation. Nous étudions les équations " bifluides " en film mince, obtenues par Bayada, Sabil et Paoli [Sab00, Pao03] à partir des équations de Stokes bifluides [NPD97]. Ainsi, un écoulement bifluide (i.e. dans lequel interviennent deux fluides de viscosité μ_l et μ_g) en film mince peut être décrit par une équation de Reynolds généralisée (décrivant le comportement de la pression) couplée à une équation de Buckley-Leverett généralisée (décrivant le comportement de la saturation du fluide référence). Les coefficients de ces équations dépendent évidemment du rapport des viscosité ainsi que de la vitesse de cisaillement. L'obtention de ces équations n'est pas rigoureuse car elle nécessite une hypothèse de graphe sur la frontière libre séparant les deux fluides. Il est donc nécessaire d'effectuer une analyse mathématique rigoureuse de ces équations, ce qui a été omis dans [Pao03] sauf lorsque le cisaillement est nul (ce qui n'est pas réaliste pour les régimes de lubrification). Nous appliquerons ce modèle à la cavitation diphasique. Ainsi, l'originalité de ce chapitre réside dans les aspects suivants développés ci-après.

- Nous réécrivons les équations limites en faisant apparaître leur structure très particulière : cette réécriture fait apparaître l'importance des effets de cisaillement qui rendent le problème non-classique.
- Nous effectuons l'analyse mathématique complète de ces équations : en particulier, nous montrons l'existence et l'unicité d'une solution (en un sens à préciser) ; par ailleurs, nous montrons que la solution de l'équation de Buckley-Leverett généralisée (incluant les effets de cisaillement), qui représente une saturation, est une fonction à valeurs dans $[0, 1]$. Ces résultats découlent directement du Chapitre 3. Rappelons que ces propriétés ne découlent pas directement du passage à la limite des équations de Stokes bifluides car ce passage a nécessité une hypothèse sur la forme de la frontière libre.
- Nous déterminons une méthode numérique permettant de simuler des écoulements bifluides. Une application directe de ces écoulements est la modélisation de la cavitation. En effet, le rapport des viscosités entre un gaz et un liquide est de l'ordre de 10^{-3}. En utilisant un jeu de données approprié, nous retrouvons avec cette approche bifluide les profils de pression-saturation issus du modèle d'Elrod-Adams.

Considérons un domaine

$$\Omega^\delta = \{(x, y) \in \mathbb{R}^2,\ 0 \le x \le L,\ 0 \le y \le \delta h(x)\}$$

avec des conditions aux limites appropriées aux régimes de lubrification (cisaillement sur la partie inférieure du domaine notamment) et rappelons dans un premier temps l'obtention de l'équation de Reynolds à partir des équations de Stokes monofluide :

1/ Equations de Stokes (voir, par exemple, [Tem84]), régissant l'écoulement d'un fluide de viscosité $\mu = \mu_l$. L'écoulement est décrit par une distribution en pression et vitesse (p^δ, u^δ).

2/ Estimations a priori, changement d'échelle et passage à la limite sur δ (approximation film mince) [BC86].

3/ Réduction du problème : équation de Reynolds [BC86] régissant le comportement de la pression ; la vitesse se déduit directement de la pression.

De manière analogue à cette démarche, Bayada, Sabil et Paoli [Pao03, Sab00] ont étudié le comportement de l'écoulement en présence de deux fluides de viscosité différente μ_l et μ_g :

1/ Equations de Stokes bifluides à frontière libre (d'après Nouri, Poupaud et Demay [NPD97]), régissant l'écoulement de deux fluides immiscibles de viscosité μ_l, μ_g. L'écoulement est décrit par une distribution en pression, vitesse et viscosité $(p^\delta, u^\delta, \mu^\delta \in \{\mu_l, \mu_g\})$.

2/ Estimations a priori, changement d'échelle et passage à la limite sur δ (approximation film mince) [Pao03, Sab00].

3/ Réduction du problème : équation de Reynolds généralisée régissant le comportement de la pression couplée à une équation de Buckley-Leverett généralisée régissant le comportement de la saturation du fluide de référence.

L'idée qui a motivé cette étude a été de considérer que l'un des fluides est une phase liquide, l'autre fluide étant une phase gaseuse, afin de simuler un mélange des deux phases, modélisant la cavitation.

■ **Section 1**

Nous présentons les équations obtenues dans [Pao03] par l'approximation " film mince " à partir du modèle bifluide de Nouri, Poupaud et Demay. Soit $\Omega =]0, L[$, $Q_T =]0, T[\times\Omega$ et $\Sigma_T =]0, T[\times\{0, L\}$. La saturation du fluide de référence s obéit à l'équation de Buckley-Leverett généralisée suivante (on se place dans un problème à une dimension d'espace x) :

$$\frac{\partial}{\partial t}\left(h(x)s(t,x)\right) + \frac{\partial}{\partial x}\left(Q_{in}(t)f(s) + v_0 h(x)g(s)\right) = 0, \quad (t,x) \in Q_T, \tag{0.16}$$

où h est la hauteur normalisée entre les surfaces, Q_{in} le débit imposé, v_0 la vitesse de cisaillement (vitesse de la surface inférieure). Les fonctions f et g dépendent évidemment du rapport des viscosités μ_g/μ_l. Dans tous les cas, f représente la contribution classique au flux de Buckley-Leverett tandis que g est une contribution non classique, induite par le cisaillement et / ou le profil variable en espace de h. En effet, f satisfait les hypothèses habituelles utilisées dans l'étude des milieux poreux ($f(0) = 0$, $f(1) = 1$ et f a un profil de type "$\mathcal{S}$-shape"). Par ailleurs, la contribution due au cisaillement possède également un profil particulier : $g(0) = g(1) = 0$. Cette équation est prise en compte avec les conditions initiales et aux limites suivantes :

$$s(0, \cdot) = s_0, \quad \text{sur } \Omega \tag{0.17}$$

$$\text{“}s = s_1\text{”}, \quad \text{sur } \Sigma_T \tag{0.18}$$

où le sens des conditions aux limites sera précisé. Notons que s_0 et s_1 sont des fonctions à valeurs dans $[0, 1]$. La pression p dans le bifluide obéit à une équation de Reynolds généralisée :

$$\frac{\partial}{\partial x}\left(A(s)\frac{h^3}{6\mu_l}\frac{\partial p}{\partial x}\right) = v_0\frac{\partial}{\partial x}\left(B(s)h\right), \quad (t, x) \in Q_T, \tag{0.19}$$

avec les conditions aux limites

$$p = 0, \quad (t, x) \in \Sigma_T. \tag{0.20}$$

Ici, A et B dépendent du rapport des viscosités.

■ **Sections 2 et 3**

Dans cets sections, nous nous intéressons à l'équation de Buckley-Leverett et nous montrons le résultat suivant :

Théorème 0.18. *Pour des données L^∞ à valeurs dans $[0, 1]$, le problème (0.16)-(0.17)-(0.18) admet une unique solution faible entropique (en un sens à préciser). C'est une fonction à valeurs dans $[0, 1]$.*

Idée de la preuve : Sous des hypothèses de régularité des données, un résultat d'existence et d'unicité peut être obtenu directement à partir des résultats de Bardos, Le Roux et Nédélec [BLRN79] ; néanmoins, cela ne permet pas d'établir que la solution est une fonction à valeurs dans $[0, 1]$. Remarquons par ailleurs que l'équation (0.16) est une loi qui est conservative par rapport à hs, ce qui permet a priori d'espérer un résultat de stabilité sur la quantité hs et non la seule quantité s. En fait, nous traitons le cas de données L^∞ et nous montrons que s est à valeurs dans $[0,1]$. Pour cela, nous effectuons le changement de variables suivant :

$$Y(x) = L\,\frac{\displaystyle\int_0^x h(t)\;dt}{\displaystyle\int_0^L h(x)\;dx}, \qquad \mathcal{T}(t) = L\,\frac{\displaystyle\int_0^t Q_{in}(s)\;ds}{\displaystyle\int_0^L h(x)\;dx}.$$

En posant alors

$$u(\mathcal{T}(t), Y(x)) = s(t,x),$$

l'équation (0.16) se réduit à

$$\text{(0.21)} \qquad \frac{\partial}{\partial \tau} u(\tau, y) + \frac{\partial}{\partial y}\left(f(u) + k(\tau, y)\; g(u)\right) = 0, \quad (\tau, y) \in (0, \widetilde{T}) \times (0,1),$$

avec $k(\tau,y) = \dfrac{v_0}{\left(Q_{in} \circ \mathcal{T}^{-1}\right)(\tau)} \left(h \circ Y^{-1}\right)(y)$ et $\widetilde{T} = \mathcal{T}(T)$.
Les données du problème deviennent

$$\text{(0.22)} \qquad u(0,y) = u_0(0,y) = s_0 \circ Y^{-1}(y), \qquad y \in]0,1[,$$

$$\text{(0.23)} \; \text{“}u(\tau,y) = u_1(\tau,y) = s_1\left(\mathcal{T}^{-1}(\tau), Y^{-1}(y)\right)\text{”}, \quad (\tau,y) \in (0,\widetilde{T}) \times \partial]0,1[.$$

Ainsi, l'équation (0.16) a été réduite en une loi de conservation scalaire par rapport à une saturation u. Les données du problème auxiliaire sont également des fonctions à valeurs dans $[0,1]$. L'existence et l'unicité d'une solution faible entropique pour ce problème auxiliaire (et par suite pour le problème initial) se déduit directement du Chapitre 3. De même, en raison des propriétés particulières de g, la solution est une fonction à valeurs dans $[0,1]$. $\square$

Nous établissons un résultat d'existence et d'unicité pour la solution (en pression) de l'équation de Reynolds généralisée (0.19)–(0.20). Nous établissons aussi des estimations a priori sur la pression en norme H^1 ou L^∞, indépendantes du rapport des viscosités ε. Par ailleurs, un résultat de convergence de Vovelle [Vov02a] établit un schéma numérique permettant de simuler ce type de problèmes, avec la prise en compte appropriée des conditions aux limites. Néanmoins, la performance du schéma dépend évidemment des données du problème. Or, pour des rapports de viscosités $\varepsilon = \mu_g/\mu_l$ faibles (rappelons que $\varepsilon \approx 10^{-3}$ pour un mélange liquide-gaz), les flux f et g de l'équation de Buckley-Leverett tendent à dégénérer. En raison d'une contrainte de type CFL, nous établissons l'ordre de grandeur des paramètres de discrétisation autorisant une résolution effective du problème :

$$\begin{array}{lcll} \text{pas d'espace :} & \Delta x & = & \mathcal{O}(\varepsilon^{1/3}), \\ \text{pas de temps :} & \Delta t & = & \mathcal{O}(\varepsilon^{2/3}). \end{array}$$

Il apparaît donc que la simulation numérique du problème pour de faibles valeurs de ε n'est pas raisonnable en termes de coûts de calculs. D'un point de vue théorique également, l'obtention d'un modèle asymptotique $\varepsilon \to 0$ constitue un enjeu important : cet aspect sera développé dans les conclusions de ce manuscrit.

- **Section 4**
  Un code de calculs a été développé afin de simuler numériquement différentes configurations. Tout d'abord, nous montrons les différences essentielles entre les régimes non-cisaillé et cisaillé, notamment du point de vue de la compréhension des conditions aux limites et du comportement du choc qui est, dans le cas de la lubrification, assimilé à la rupture du film fluide. Puis, nous effectuons des tests permettant de comparer le modèle d'Elrod-Adams avec le modèle Buckley-Leverett / Reynolds, pour des régimes de fonctionnement identiques. Rappelons que le modèle d'Elrod-Adams introduit l'idée que le film lubrifié n'est que partiel dans les zones cavitées. Mais il ne donne aucune indication sur la répartition des phases à l'intérieur de cette zone. On peut ainsi penser (voir FIG.10) que le lubrifiant liquide est attaché à la paroi mobile (inférieure) ou, inversement, qu'il est attaché à la paroi mobile (supérieure), ou encore que deux couches de lubrifiant liquide entourent une couche gazeuse (multi-couches) ou une structure en bulles. Le modèle bifluide que nous avons développé est pertinent dans les deux premiers cas, qui respectent l'hypothèse sur la forme de la frontière libre. Nous comparons la solution du modèle d'Elrod-Adams à celle du modèle bifluide (pour $\mu_g/\mu_l = 10^{-3}$). Lorsque l'on suppose que le lubrifiant est rattaché à la paroi mobile, nous obtenons une solution dont les propriétés sont très proches de celle d'Elrod-Adams, ce qui permet de valider certains aspects du modèle bifluide.

0.2.5 Résumé de l'Appendice A

Homogénéisation du problème de la digue : prise en compte des hétérogénéités de la perméabilité

Nous étudions ici l'homogénéisation du problème de la digue, tel qu'il a été formulé par Brezis, Kinderlehrer et Stampacchia [BKS78] et Alt [Alt79]. Le domaine d'étude est un milieu poreux de perméabilité variable. L'objectif est de déterminer la localisation

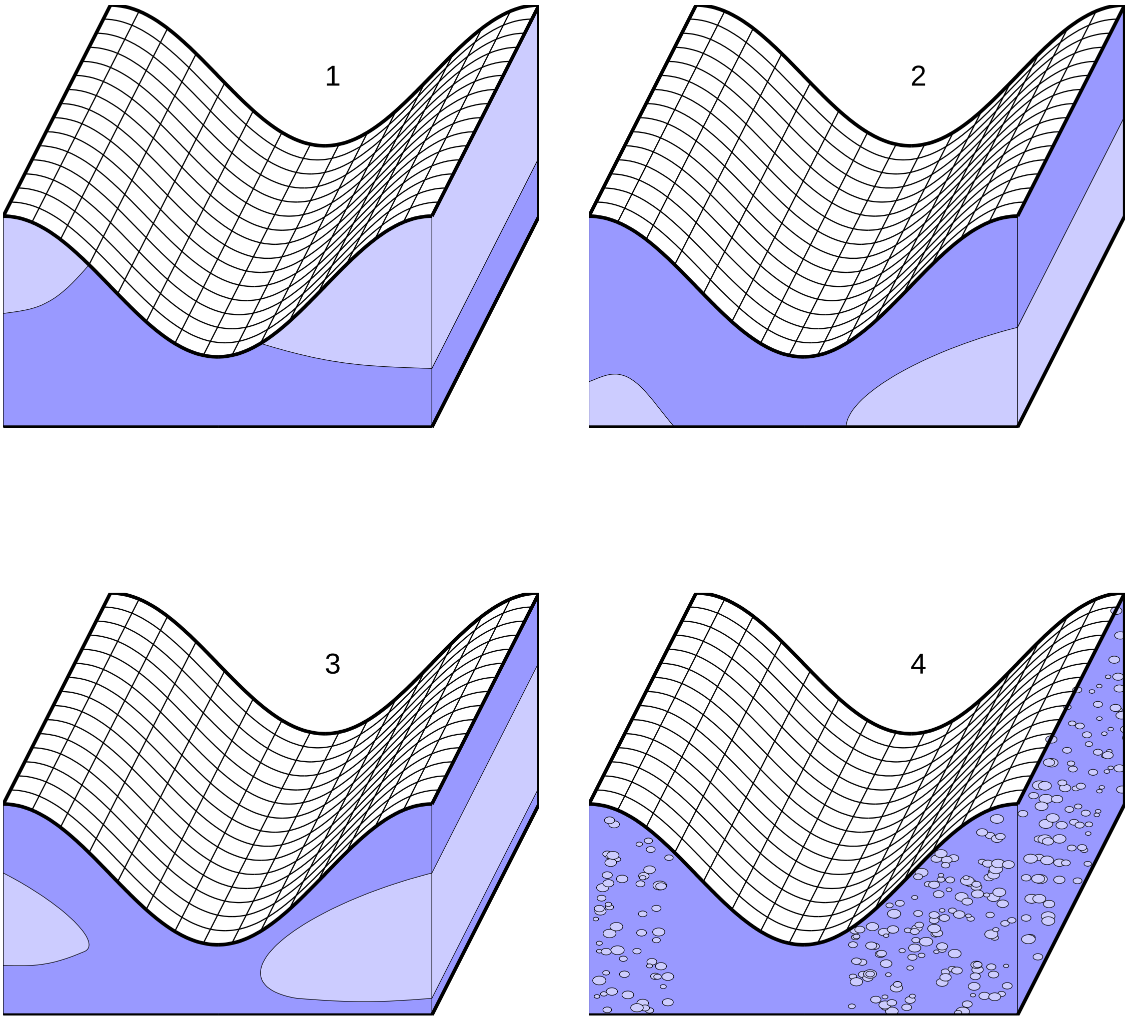


Figure 10. Différentes interprétations de la cavitation : (1) phase gaseuse fixée à la surface immobile, (2) phase gaseuse fixée à la surface mobile, (3) multi-couches, (4) bulles

des zones mouillées et sèches (ou partiellement mouillées) ainsi que la distribution de pression et la vitesse de filtration du liquide à l'intérieur de la digue. L'homogénéisation est motivée par la prise en compte des variations de la perméabilité, notamment selon un profil stratifié. Nous fournissons ici les coefficients homogénéisés pour différents types de strates. Ces résultats généralisent, d'un certain point de vue, ceux obtenus par Rodrigues [Rod84].

La formulation du problème est parfaitement similaire à celle du problème de lubrification hydrodynamique, à quelques différences près, celles-ci n'étant pas fondamentales du point de vue de l'analyse mathématique. En conséquence, nous établissons les résultats sans démonstration qui, de façon évidente, sont des adaptations immédiates de celles qui sont développées au Chapitre 1. En particulier, nous explicitons les coefficients homogénéisés dans le cas de strates verticales, horizontales et obliques. Notons que, dans le cas général, les problèmes d'anisotropie de la saturation existent (comme dans le problème de lubrification). De même, la méthode de résolution dans le cas de strates obliques nécessite une stratégie identique à celle qui a été utilisée dans le problème de lubrification (avec rugosités obliques) : dans ce cas, un changement de coordonnées nous renvoie à un problème de la digue " généralisé ", au sens où il fait intervenir un terme de gravité " oblique " et non plus " vertical ".

0.2.6 Résumé des Appendices B et C

Comportement asymptotique d'un modèle conservatif de type Reynolds pour la prise en compte des rugosités de surface

Ces appendices sont constitués de deux articles de mécanique. Nous reprenons certains aspects développés dans les Chapitres 1 et 2 en développant certains points qui sont importants du point de vue des applications. En particulier, on s'intéresse aux aspects hydrodynamiques et élastohydrodynamiques :

- *Problème hydrodynamique :*
  Nous utilisons le formalisme des développements asymptotiques, validés de manière rigoureuse par le Chapitre 1, afin d'obtenir les équations limites. Dans la partie numérique, on s'intéresse à la simulation d'écoulements lubrifiés pour des profils de rugosités 2D, et non plus seulement purement longitudinales ou purement transverses ; en particulier, on s'intéresse au cas des rugosités obliques et on montre que, au moins numériquement et dans le cadre du mécanisme étudié (un patin rugueux, dont les deux surfaces moyennes en regard sont parallèles), la solution avec saturation isotrope est pertinente. Autrement dit, on observe la convergence de la pression vers la pression du problème homogénéisé avec saturation isotrope. On s'intéresse également à la cavitation interaspérités qui, dans le cas du patin, est engendrée uniquement par les rugosités ; on retrouve numériquement les résultats sur le correcteur exposés par Harp et Salant [HS01] dans un cadre non déterministe.

- *Problème élastohydrodynamique :*
  En reprenant le développement asymptotique issu du cas hydrodynamique, nous obtenons les équations élastohydrodynamiques limites. Les tests numériques permettent de mettre en évidence l'influence des rugosités (notamment le lien entre le minimum d'épaisseur du mécanisme et l'amplitude des rugosités) sur la charge supportée par le mécanisme et l'influence de la piézoviscosité sur les pics de pression.

Bibliographie

[AC94] S. J. Alvarez et J. Carrillo. A free boundary problem in theory of lubrication. *Comm. Partial Differential Equations*, 19(11-12):1743–1761, 1994.

[All92] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.

[Alt79] H. W. Alt. Strömungen durch inhomogene poröse Medien mit freiem Rand. *J. Reine Angew. Math.*, 305:89–115, 1979.

[Alt81] H. W. Alt. Numerical solution of steady-state porous flow free boundary problems. *Numer. Math.*, 36(1):73–98, 1980/81.

[Alv86] S. J. Alvarez. Problemas de frontera libre en teoria de lubrificacion. *Tesis de doctorado, Universidad Complutense de Madrid, Spain*, 1986.

[AO03] S. J. Alvarez et R. Oujja. On the uniqueness of the solution of an evolution free boundary problem in theory of lubrication. *Nonlinear Anal.*, 54(5):845–872, 2003.

[Bar93] C. Barus. Isothermals, isopiestics and isometrics relative to viscosity. *American Journal of Science*, 45:87–96, 1893.

[Bay72] G. Bayada. Inéquations variationnelles elliptiques aux conditions aux limites périodiques. Applications à la résolution de l'équation de Reynolds. *Thèse de 3ème cycle, Université de Lyon*, 1972.

[BC82] G. BAYADA et M. CHAMBAT. Nonlinear variational formulation for a cavitation problem in lubrication. *J. Math. Anal. Appl.*, 90(2):286–298, 1982.

[BC84] G. BAYADA et M. CHAMBAT. Existence and uniqueness for a lubrication problem with nonregular conditions on the free boundary. *Boll. Un. Mat. Ital. B (6)*, 3(2):543–557, 1984.

[BC86] G. BAYADA et M. CHAMBAT. The transition between the Stokes equations and the Reynolds equation: a mathematical proof. *Appl. Math. Optim.*, 14(1):73–93, 1986.

[BC83] G. BAYADA et M. CHAMBAT. Analysis of a free boundary problem in partial lubrication. *Quart. Appl. Math.*, 40(4):369–375, 1982/83.

[BCV98] G. BAYADA, M. CHAMBAT, et C. VÁZQUEZ. Characteristics method for the formulation and computation of a free boundary cavitation problem. *J. Comput. Appl. Math.*, 98(2):191–212, 1998.

[BD87] A. BERMÚDEZ et J. DURANY. La méthode des caractéristiques pour les problèmes de convection-diffusion stationnaires. *RAIRO Modél. Math. Anal. Numér.*, 21(1):7–26, 1987.

[BD89] A. BERMÚDEZ et J. DURANY. Numerical solution of cavitation problems in lubrication. Dans *Proceedings of the Eighth International Conference on Computing Methods in Applied Sciences and Engineering (Versailles, 1987)*, volume 75 (1-3), pages 457–466, 1989.

[BEATV96] G. BAYADA, M. EL ALAOUI TALIBI, et C. VÁZQUEZ. Existence of solutions for elastohydrodynamic piezoviscous lubrication problems with a new model of cavitation. *European J. Appl. Math.*, 7(1):63–73, 1996.

[BKS78] H. BREZIS, D. KINDERLEHRER, et G. STAMPACCHIA. Sur une nouvelle formulation du problème de l'écoulement à travers une digue. *C. R. Acad. Sci. Paris Sér. A-B*, 287(9):A711–A714, 1978.

[BLRN79] C. BARDOS, A. Y. LE ROUX, et J.-C. NÉDÉLEC. First order quasilinear equations with boundary conditions. *Comm. Partial Differential Equations*, 4(9):1017–1034, 1979.

[BM81] A. BERMÚDEZ et C. MORENO. Duality methods for solving variational inequalities. *Comput. Math. Appl.*, 7(1):43–58, 1981.

[BMV05a] G. Bayada, S. Martin, et C. Vázquez. About a generalized Buckley-Leverett equation and lubrication multifluid flow. *Preprint Institut Camille Jordan CNRS-UMR 5208 (submitted for publication)*, 2005.

[BMV05b] G. Bayada, S. Martin, et C. Vázquez. An average flow model of the Reynolds roughness including a mass-flow preserving cavitation model. *ASME J. Tribol.*, 127(4):793–802, 2005.

[BMV05c] G. Bayada, S. Martin, et C. Vázquez. Effets d'anisotropie par homogénéisation dans un problème à frontière libre. *C. R. Math. Acad. Sci. Paris*, 340(7):541–546, 2005.

[BMV05d] G. Bayada, S. Martin, et C. Vázquez. Homogenization of a nonlocal elastohydrodynamic lubrication problem: a new free boundary model. *Math. Mod. Meth. Appl. Sci.*, 15(12):1923–1956, 2005.

[BMV05e] G. Bayada, S. Martin, et C. Vázquez. Micro-roughness effects in (elasto)hydrodynamic lubrication including a mass-flow preserving cavitation model. *Preprint Institut Camille Jordan CNRS-UMR 5208 (submitted for publication)*, 2005.

[BMV05f] G. Bayada, S. Martin, et C. Vázquez. Two-scale homogenization of a hydrodynamic Elrod-Adams model. *Asymptot. Anal.*, 44(1-2):75–110, 2005.

[Car99] J. Carrillo. Entropy solutions for nonlinear degenerate problems. *Arch. Ration. Mech. Anal.*, 147(4):269–361, 1999.

[CC81] J. Carrillo et M. Chipot. Sur l'unicité de la solution du problème de l'écoulement à travers une digue. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(3):191–194, 1981.

[CDG02] D. Cioranescu, A. Damlamian, et G. Griso. Periodic unfolding and homogenization. *C. R. Math. Acad. Sci. Paris*, 335(1):99–104, 2002.

[Chr41] D. G. Christopherson. A new mathematical method for the solution of film lubrication problems. *Inst. Mech. Engrs. J. Proc.*, 146:126–135, 1941.

[Cim77] G. Cimatti. On a problem of the theory of lubrication governed by a variational inequality. *Appl. Math. Optim.*, 3(2/3):227–242, 1976/77.

[CM78] G. Cimatti et O. Menchi. On the numerical solution of a variational inequality connected with the hydrodynamic lubrication of a complete journal bearing. *Calcolo*, 15(3):249–258, 1978.

[DGT75] D. Dowson, M. Godet, et C. M. Taylor, éditeurs. *Cavitation and Related Phenomena in Lubrication.* Mechanical Engineering Publications for the Institute of Tribology, Leeds University and the Institut National des Sciences Appliquées de Lyon, 1975.

[DGV96a] J. Durany, G. García, et C. Vázquez. A mixed Dirichlet-Neumann problem for a nonlinear Reynolds equation in elastohydrodynamic piezoviscous lubrication. *Proc. Edinburgh Math. Soc. (2)*, 39(1):151–162, 1996.

[DGV96b] J. Durany, G. García, et C. Vázquez. Numerical computation of free boundary problems in elastohydrodynamic lubrication. *Appl. Math. Modelling*, 20:104–113, 1996.

[DGV02] J. Durany, G. García, et C. Vázquez. Numerical simulation of a lubricated Hertzian contact problem under imposed load. *Finite Elem. Anal. Des.*, 38(7):645–658, 2002.

[Dow63] G. I. Dowson. Cavitation of a viscous fluid in narrow passages. *J. Fluid Mech.*, 16:595–619, 1963.

[EA75] H. G. Elrod et M. L. Adams. A computer program for cavitation. *Cavitation and related phenomena in lubrication - Proceedings - Mech. Eng. Publ. ltd*, pages 37–42, 1975.

[Elr60] H. G. Elrod. A derivation of the basic equations for hydrodynamic lubrication with a fluid having constant properties. *Quart. Appl. Math.*, 17:349–359, 1959/1960.

[Elr81] H. G. Elrod. A cavitation algorithm. *ASME J. Lubrication Technol.*, 103:350–354, 1981.

[FJ57] L. Floberg et B. Jakobsson. The finite journal bearing considering vaporization. *Transactions of Chalmers University of Technology, Gutenberg, Sweden*, 190, 1957.

[Flo73] L. Floberg. Lubrication of two rotating cylinders at variable lubricant supply with reference to the tensile strength of the liquid lubricant. *ASME J. Lub. Technol.*, 95:155–165, 1973.

[G̈21] L. K. R. Gümbel. Vergleich der Ergebrusse der rectinerischen Behandling des Lagerschmierangsproblem mit neueren Veisuchsergebrussen. *Monatsbl. Berliner Bez. Ver. Dtsch. Ing.*, pages 125–128, 1921.

[GLT76] R. Glowinski, J.-L. Lions, et R. Trémolières. *Analyse numérique des inéquations variationnelles. Tome 1.* Dunod, Paris, 1976. Théorie générale premières applications, Méthodes Mathématiques de l'Informatique, 5.

[HB25] W. Hardy et I. Bircumshaw. Boundary lubrication: plane surfaces and the limitations of Amontons law. *Proc. R. Soc. London*, 108:1–27, 1925.

[Her82] H. Hertz. Über die Berührung fester elasticher Körper. *J. Reine und Angew. Math.*, 92:156–192, 1882.

[HS01] S. H. Harp et R. F. Salant. An average flow model of rough surface lubrication with inter-asperity cavitation. *ASME J. Tribol.*, 123:134–143, 2001.

[Kru70] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[KS00] D. Kinderlehrer et G. Stampacchia. *An introduction to variational inequalities and their applications*, volume 31 de *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. Reprint of the 1980 original.

[Lam93] H. Lamb. *Hydrodynamics*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, sixth édition, 1993. With a foreword by R. A. Caflisch [Russel E. Caflisch].

[LM74] A. Laratta et O. Menchi. Approssimazione della soluzione di una disequazione variazionale. Applicazione ad un problema di frontiera libera. *Calcolo*, 11:243–267, 1974.

[Loh70] J. Lohou. Hydrodynamique des joints d'étanchéité du type radial. *Thèse de doctorat, Université de Lyon*, 1970.

[Mar68] P. Marzulli. Risoluzione alle differenze di equazioni alle derivate parziali di tipo ellitico con condizioni su un contorno libero. *Calcolo*, 5, 1968.

[Mar05] S. Martin. First order quasilinear equations with boundary conditions in the L^∞ framework. *Preprint Institut Camille Jordan CNRS-UMR 5208 (submitted for publication)*, 2005.

[Mey81] G. H. Meyer. An analysis of the method of lines for the Reynolds equation in hydrodynamic lubrication. *SIAM J. Numer. Anal.*, 18(1):165–177, 1981.

[Mic05] A. G. M. Michell. The lubrication of plane surfaces. *Z. Math. Phys.*, 52:123–137, 1905.

[MP86] L. D. Marini et P. Pietra. Fixed-point algorithms for stationary flow in porous media. *Comput. Methods Appl. Mech. Engrg.*, 56(1):17–45, 1986.

[Naz90] S. A. Nazarov. Asymptotic solution of the Navier-Stokes problem on the flow of a thin layer of fluid. *Sibirsk. Mat. Zh.*, 31(2):131–144, 1990.

[NPD97] A. Nouri, F. Poupaud, et Y. Demay. An existence theorem for the multi-fluid Stokes problem. *Quart. Appl. Math.*, 55(3):421–435, 1997.

[Ols65] K. O. Olsson. Cavitation in dynamically loaded bearing. *Transactions of Chalmers University of Technology, Guthenberg, Sweden*, 308, 1965.

[Ott96] F. Otto. Initial-boundary value problem for a scalar conservation law. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(8):729–734, 1996.

[Pan80] C. H. T. Pan. An improved short bearing analysis for the submerged operation of plain journal bearings and squeeze-film dampers. *ASME J. Lubrication Technol.*, 102:320–331, 1980.

[Pao03] L. Paoli. Asymptotic behavior of a two fluid flow in a thin domain: from Stokes equations to Buckley-Leverett equation and Reynolds law. *Asymptot. Anal.*, 34(2):93–120, 2003.

[PI81] C. H. T. Pan et R. A. Ibrahim. Cavitation in a short bearing with pressurized lubricant supply. *ASME J. Lubrication Technol.*, 103:337–346, 1981.

[Rey86] O. Reynolds. On the theory of lubrication and its application to Mr Beauchamp tower's experiments, including an experimental determination of the viscosity of olive oil. *Phil. Trans. Roy. Soc.*, A 117:157–234, 1886.

[Rod84] J.-F. Rodrigues. Some remarks on the homogenization of the dam problem. *Manuscripta Math.*, 46(1-3):65–82, 1984.

[Sab00] J. Sabil. Etude théorique et numérique d'un écoulement diphasique non-miscible et incompressible dans un domaine mince. *Rapport de Recherche du laboratoire MAPLY CNRS-UMR 5585 (Stage de D.E.A.)*, 2000.

[Ser96] D. Serre. *Systèmes de lois de conservation. II.* Fondations. [Foundations]. Diderot Editeur, Paris, 1996. Structures géométriques, oscillation et problèmes mixtes. [Geometric structures, oscillation and mixed problems].

[Som04] A. Sommerfeld. Zur hydrodynamischen Theorie der Schmiermittelreibung. *Z. Math. Phys.*, 40:97–155, 1904.

[Sta72] G. Stampacchia. On a problem of numerical analysis connected with the theory of variational inequalities. Dans *Symposia Mathematica, Vol. X (Convegno di Analisi Numerica, INDAM, Rome, 1972)*, pages 281–293. Academic Press, London, 1972.

[Sti33] W. Stieber. Das Schwimmlager. *Ver. Dtsch. Ing. (Berlin)*, 1933.

[Swi32] H. W. Swift. The stability of lubricating films in journal bearings. *Proc. Inst. Civil Engrs. (London)*, 233:267–288, 1932.

[Tem84] R. Temam. *Navier-Stokes equations.* Studies in Mathematics and its Applications, 2. North-Holland, Amsterdam, 1984. Theory and numerical analysis. With an appendix by F. Thomasset (Third edition).

[Váz94] C. Vázquez. Existence and uniqueness of solution for a lubrication problem with cavitation in a journal bearing with axial supply. *Adv. Math. Sci. Appl.*, 4(2):313–331, 1994.

[Váz96] C. Vázquez. Etude mathématique d'un modèle de cavitation en lubrification hydrodynamique et élastohydrodynamique. *Thèse de doctorat de l'Université Claude Bernard - Lyon I*, 1996.

[Ven92] C. H. Venner. Higher order multilevel solvers for the EHL line and point contact problem. *ASME J. Tribol.*, 116:741–750, 1992.

[VL93] C. H. Venner et A. A. Lubrecht. Multilevel solvers for integro and integro-differential problems in contact mechanics and lubrication. Dans *Proceedings 4th European Multigrid Conference (Amsterdam, The Netherlands)*, pages 111–127. Editors P.W. Hemker and P Wesseling, Birkhauser Verlag, 1993.

[VL00] C. H. Venner et A. A. Lubrecht. *Multi-level methods in lubrication.* Tribology and Interface Engineering Series, 37. Elsevier, 2000.

[Vov02a] J. Vovelle. Convergence of finite volume monotone schemes for scalar conservation laws on bounded domains. *Numer. Math.*, 90(3):563–596, 2002.

[Vov02b] J. Vovelle. Prise en compte des conditions aux limites dans les équations hyperboliques non-linéaires. *Thèse de doctorat de l'Université d'Aix-Marseille I*, 2002.

[VtNB90] C. H. Venner, W. E. ten Napel, , et R. Bosma. Advanced multilevel solution of the EHL line contact problem. *ASME J. Tribol.*, 112:426–432, 1990.

[Wan50] G. H. Wannier. A contribution to the hydrodynamics of lubrication. *Quart. Appl. Math.*, 8:1–32, 1950.

1

Two-scale homogenization of the (hydrodynamic) Elrod-Adams model

Note parue aux
Comptes Rendus de l'Académie des Sciences, Série Mathématique

Article paru dans
Asymptotic Analysis

Abstract *The present chapter deals with the analysis and homogenization of a lubrication problem, via two-scale convergence. We study in particular the Elrod-Adams problem with highly oscillating roughness effects.*

1.0 Statement of the problem

Cylindrical thin film bearings are commonly used for load support of rotating machinery. Fluid film bearings also introduce viscous damping that aids in reducing the amplitude of vibrations in operating machinery. A plain cylindrical journal bearing is made of an inner rotating cylinder and an outer cylinder. The two cylinders are closely spaced and the annular gap between the two cylinders is filled with some lubricant. The radial clearance is very small, typically $\Delta r/r = 10^{-3}$ for oil lubricated bearings. The smallness of this ratio allows for a Cartesian coordinate to be located on the bearing surface. Thus, the Reynolds equation has been used for a long time to describe the behaviour of a viscous flow between two close surfaces in relative motion (see [Rey86, Rey99] for historical references). The transition of the Stokes equation to the Reynolds equation has been proven by Bayada and Chambat in [BC86]. In dimensionless coordinates, it can be written as

$$\nabla \cdot \Big(h^3 \nabla p\Big) = \frac{\partial}{\partial x_1}\Big(h\Big),$$

where p is the pressure distribution, and h the height between the two surfaces.

Nevertheless, this modelling does not take into account cavitation phenomena: cavitation is defined as the rupture of the continuous film due to the formation of air bubbles and makes the Reynolds equation no longer valid in the cavitation area. In order to make it possible, various models have been used, the most popular perhaps being variational inequalities which have a strong mathematical basis but lack physical evidence. Thus, we use the Elrod-Adams model, which introduces the hypothesis that the cavitation region is a fluid-air mixture and an additional unknown θ (the saturation of fluid in the mixture) (see [CC83, CE70, CE71, EA75]). The model includes a modified Reynolds equation, here referred to as *exact Reynolds equation with cavitation* (see problem $(\mathcal{P}_\theta)$ in the next section). From a mathematical point of view, the problem can be simplified using a *penalized Reynolds equation with cavitation* (see problem $(\mathcal{P}_\eta)$ in the next section).

The homogenization process for lubrication problems is mainly related to the roughness of the surfaces. Let us mention that the Reynolds equation is still valid as long as $\varepsilon/\sigma \gg 1$, ε being a small parameter describing the roughness spacing, and σ being the film thickness order (assumed to be small too) (see [BC88] for details). The study of surface roughness effects in lubrication has gained an increasing attention from 1960 since it was thought to be an explanation for the unexpected load support in bearings.

Several methods have been used in order to study roughness effects in lubrication, the most popular perhaps being the flow factor method (see [PC78, PC79, Tri83]), which is based on a formulation that is close to the initial one, only modified by flow factors related to anistropic and microscopic effects.

So far this procedure has been used either by considering that no cavitation phenomena occur or using variational inequation models. Let us mention that the homogenization of cavitation models using variational inequalities has been studied in [BF89]. Recently many papers have discussed cavitation phenomena coupled with roughness effects, in mechanical engineering:

- A generalized computational formulation, by Shi and Salant [SS00], has been applied to the rotary lip seal and used to predict the performance characteristics over a range of shaft speeds.

- Interasperity cavitation has been studied in particular by Harp and Salant in [HS01] in order to derive a modified Reynolds equation with flow factors describing roughness effects and macroscopic cavitation.

- Modelling of cavitation has been pointed out in particular by Van Odyck and Venner in [VOV03] in order to discuss the validity of the Elrod-Adams model and the formation of air bubbles leading to cavitation phenomena.

The above papers are based on averaging methods taking into account statistic roughness and are mainly heuristic. Our purpose, in the present chapter, is to study in a rigorous way the limit of a three dimensional Stokes flow between two close rough surfaces using a double scale asymptotic expansion analysis (see for instance [BCF88]) in the Elrod-Adams model.

The chapter is organized as follows:

- Section 1.1 is devoted to the mathematical formulation of the lubrication problem: we briefly present the exact Elrod-Adams problem along with its penalized version. We also give the existence and uniqueness results corresponding to each problem. For this, we use a well-known penalization method to get the existence result. Uniqueness of the pressure is obtained using the doubling variable method of Kružkov, which has been extended by Carrillo to the dam problem.

- Section 1.2 deals with the homogenization process: after some preliminaries on the two-scale technique, we first establish an uncomplete form of the homogenized problem in which an additional term in the direction perpendicular to the flow but also anisotropic phenomena on the saturation appear. In order to complete the homogenized problem, we introduce additional assumptions that lead us to consider particular but realistic cases: considering a separation of the microvariables on the gaps allows us to completely solve the difficulties previously mentioned; then, taking into account oblique roughness, we show that we obtain an intermediary case between the uncomplete problem (general case) and the complete problem (with the separation of the microvariables).

- Section 1.3 presents the numerical method and results which illustrate the main theorems established in the previous sections: we study longitudinal and transverse roughness cases.

1.1 Mathematical formulation

1.1.1 The lubrication problem

The dimensionless domain is denoted $\Omega =]0, 2\pi[\times]0, 1[$ and we suppose that the following assumptions are satisfied:

Assumption 1. *$h \in C^1(\Omega)$ is $2\pi x_1$ periodic and satisfies: $0 < h_0 \leq h \leq h_1$ on Ω.*

Assumption 2. *$p_a \in C^1(0, 2\pi)$ is a 2π periodic non-negative function.*

Now let us introduce the Elrod-Adams model taking into account cavitation phenomena. We introduce an *exact* problem and a *penalized* problem.

(i) *Exact Reynolds problem* - The strong formulation of the problem is given by the following set of equations:

$$\begin{cases} -\nabla \cdot \Big(h^3(x)\nabla p(x)\Big) = -\dfrac{\partial}{\partial x_1}\Big(\theta(x)\ h(x)\Big), & x \in \Omega \\ p(x) \geq 0, \quad p(x)\ (1 - \theta(x)) = 0, \quad 0 \leq \theta(x) \leq 1, & x \in \Omega \end{cases}$$

with the following boundary conditions:

$$\begin{array}{ll} p = 0 \text{ on } \Gamma_0 \text{ and } p = p_a \text{ on } \Gamma_a, & \text{(Dirichlet conditions)} \\ \theta h - h^3 \dfrac{\partial p}{\partial x_1} \text{ and } p \text{ are } 2\pi x_1 \text{ periodic}, & \text{(periodic conditions)} \end{array}$$

where $\theta(x)$ is the normalized height of fluid between the two surfaces. The boundaries Γ_0 and Γ_a are given on FIG.1.1. These boundary conditions are linked with a specific but wide type of bearings: journal bearings with a pressure imposed on the top and at the bottom. However, other boundary conditions can be considered.

Introducing the functional spaces

$$\begin{aligned} V_a &= \Big\{\phi \in H^1(\Omega),\ \phi \text{ is } 2\pi x_1 \text{ periodic},\ \phi_{|\Gamma_0} = 0,\ \phi_{|\Gamma_a} = p_a\Big\}, \\ V_0 &= \Big\{\phi \in H^1(\Omega),\ \phi \text{ is } 2\pi x_1 \text{ periodic},\ \phi_{|\Gamma_0} = 0,\ \phi_{|\Gamma_a} = 0\Big\}, \end{aligned}$$

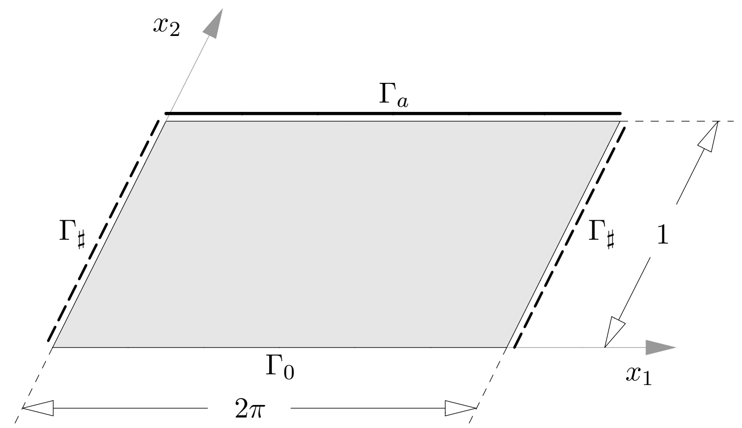


Figure 1.1. Normalized lubrication domain (with supply pressure)

the earlier problem can be formulated under a weak form as[1]

$$(\mathcal{P}_\theta)\begin{cases} \textit{Find } (p,\theta) \in V_a \times L^\infty(\Omega) \textit{ such that:} \\ \displaystyle\int_\Omega h^3\ \nabla p\ \nabla\phi = \int_\Omega \theta\ h\ \frac{\partial\phi}{\partial x_1}, \quad \forall\ \phi \in V_0, \\ p \geq 0, \quad p\ (1-\theta) = 0, \quad 0 \leq \theta \leq 1, \quad \textit{a.e.}, \end{cases}$$

(*ii*) *Penalized Reynolds problem* - In the penalized problem, an approximate relation between p and θ is used. Defining the function

$$H_\eta(z) = \begin{cases} 0, & \text{if } z < 0, \\ z/\eta, & \text{if } 0 \leq z \leq \eta, \\ 1, & \text{if } z > 1, \end{cases}$$

the weak formulation of the problem is given by

$$(\mathcal{P}_\eta)\begin{cases} \textit{Find } p_\eta \in V_a, \textit{ such that:} \\ \displaystyle\int_\Omega h^3\ \nabla p_\eta\ \nabla\phi = \int_\Omega H_\eta(p_\eta)\ h\ \frac{\partial\phi}{\partial x_1}, \quad \forall\ \phi \in V_0, \\ p_\eta \geq 0, \quad \textit{a.e.} \end{cases}$$

[1]Notice that the following formulation can be used:

$$(\mathcal{P}'_\theta)\begin{cases} \textit{Find } (\hat{p},\theta) \in V_0 \times L^\infty(\Omega) \textit{ such that:} \\ \displaystyle\int_\Omega h^3\ \nabla\hat{p}\ \nabla\phi = \int_\Omega \theta\ h\ \frac{\partial\phi}{\partial x_1} - \int_\Omega h^3\ \nabla\hat{p}_a\ \nabla\phi, \quad \forall\ \phi \in V_0, \\ \hat{p} \geq -\hat{p}_a, \quad (\hat{p}+\hat{p}_a)\ (1-\theta) = 0, \quad 0 \leq \theta \leq 1, \quad \textit{a.e.}, \end{cases}$$

Here, $\hat{p}_a$ is some $2\pi x_1$ periodic (regular) function such that $\hat{p}_a = p_a$ on Γ_a and $\hat{p}_a = 0$ on Γ_0 (for instance, $\hat{p}_a(x) = p_a(x_1)x_2$). The function $\hat{p}$ is called the reduced pressure. This formulation is equivalent to the one corresponding to problem $(\mathcal{P}_\theta)$. Here, the unknown (reduced) pressure belongs to the same functional space than the test functions. The real pressure should be read as $p = \hat{p} + \hat{p}_a \in V_a$.

Hence, $H_\eta(p_\eta)$ plays the role of the saturation function.

Let us mention that, in many ways, the lubrication problem is close to the dam problem. The dam problem has first been stated using variational inequalities (see [Bai80, BCMP73, BF77, Ben74]). But this approach is only possible for dams with vertical walls (typically rectangular dams). The formulation of the dam problem for domains with general shapes has been introduced by Brezis, Kinderlehrer, Stampacchia [BKS78] and Alt [Alt79]. Introducing the permeability of the porous medium, denoted k, the formulation is based on Darcy's law ([Dar56] for historical references). The basic problem is to find the pressure p and the fluid saturation θ in the domain. The main differences with the lubrication problem lie in the flow direction (x_1 in the lubrication problem, x_2 in the dam problem) and an additive sign condition on the fluid flow in the dam problem, designed to eliminate the non physical solutions and meaning that no water flows into the dam through the boundary in contact with the open air. Homogenization of the dam problem using the Γ-convergence has been partially studied by Rodrigues (see [Rod84] and related references). Additional (new) results are presented in Appendix A.

1.1.2 Existence and uniqueness results for $(\mathcal{P}_\eta)$

Let $(\mathcal{P}_\eta^n)$ be the auxiliary problem defined by

$$(\mathcal{P}_\eta^n)\left\{\begin{array}{l}\text{Find } p_\eta^n \in V_a \text{ such that, } p_\eta^{n-1} \in V_a \text{ being given,}\\ \displaystyle\int_\Omega h^3\ \nabla p_\eta^n\ \nabla\phi = \int_\Omega H_\eta(p_\eta^{n-1})\ h\ \frac{\partial\phi}{\partial x_1}, \quad \forall\ \phi \in V_0.\end{array}\right.$$

Lemma 1.1. *Under Assumptions 1 and 2, problem $(\mathcal{P}_\eta^n)$ admits a unique solution p_n^η. Moreover, one has the following estimates:*

$$\left\|p_n^\eta\right\|_{H^1(\Omega)} \leq C,$$

where C does not depend on n.

Proof. Equivalently, with $q_\eta^n(x_1,x_2) = p_\eta^n(x_1,x_2) - p_a(x_1)\psi(x)$ (with $\psi(x) = x_2$ for example), one has to find $q_\eta^n \in V_0$ such that

$$\int_\Omega h^3\ \nabla q_\eta^n\ \nabla\phi = \int_\Omega H_\eta(p_\eta^{n-1})\ h\ \frac{\partial\phi}{\partial x_1} - \int_\Omega h^3\ \nabla(p_a\psi)\ \nabla\phi, \quad \forall\ \phi \in V_0.$$

Existence and uniqueness are consequences of Lax-Milgram's theorem. Estimates are obtained using q_η^n as a test function and Cauchy-Schwartz inequality, trace theorem and Poincaré-Friedrichs inequality. □

Theorem 1.2. *Under Assumptions 1 and 2, problem $(\mathcal{P}_\eta)$ admits a unique solution p^η.*

Proof.

■ The existence of a solution is obtained by studying the behaviour of p_η^n when n goes to $+\infty$. By estimates of Lemma 1.1, there exists $p_\eta \in H^1(\Omega)$ such that, up to a subsequence,

$$p_\eta^n \rightharpoonup p_\eta, \quad \text{in } H^1(\Omega).$$

Consequently,

$$\int_\Omega h^3 \; \nabla p_\eta^n \; \nabla \phi \longrightarrow \int_\Omega h^3 \; \nabla p_\eta \; \nabla \phi,$$

for every $\phi \in V_0$.

As $H^1(\Omega) \hookrightarrow L^2(\Omega)$ with compact injection and H_η is Lipschitz continuous, one has

$$\int_\Omega H_\eta(p_\eta^n) \; h \; \frac{\partial \phi}{\partial x_1} \longrightarrow \int_\Omega H_\eta(p_\eta) \; h \; \frac{\partial \phi}{\partial x_1},$$

for every $\phi \in V_0$. Then one has:

$$\int_\Omega h^3 \; \nabla p_\eta \; \nabla \phi = \int_\Omega H_\eta(p_\eta) \; h \; \frac{\partial \phi}{\partial x_1}, \quad \forall \phi \in V_0. \tag{1.1}$$

Moreover, by Theorem III.9 of [Bre83],

$$p_\eta \in V_a. \tag{1.2}$$

From Equations (1.1) and (1.2), we deduce that p_η is a solution of $(\mathcal{P}_\eta)$.

■ The positivity of solutions is obtained by rewriting p_η as $p_\eta = p_\eta^+ - p_\eta^-$ with

$$\begin{array}{rcl} p_\eta^+ & = & \max(p_\eta, 0), \\ p_\eta^- & = - & \min(p_\eta, 0). \end{array}$$

It can be proved that $p_\eta^- \in V_0$. Using p_η^- as a test function in the variational formulation (1.1), one has

$$\int_\Omega h^3 \; \left| \nabla p_\eta^- \right|^2 = 0.$$

Then $p_\eta^- = 0$ a.e. and $p_\eta \geq 0$ a.e.

■ The uniqueness of the solution is obtained using a particular test function (following an idea developped in [BB93]). Let p_1 and p_2 be two solutions of $(\mathcal{P}_\eta)$. Then $q = p_1 - p_2$

satisfies:

$$\int_\Omega h^3 \ \nabla q \ \nabla\phi = \int_\Omega \Big(H_\eta(p_1) - H_\eta(p_2)\Big) \ h \ \frac{\partial\phi}{\partial x_1}, \quad \forall \ \phi \in V_0. \tag{1.3}$$

We consider the test function $\phi = f_\delta(q)$, where f_δ is defined with the usual notation for the positive part of a function by

$$f_\delta(x) = \begin{cases} \Big(1 - \dfrac{\delta}{x}\Big)^+, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

Since f_δ is Lipschitz continuous, $\phi = f_\delta(q) \in V_0$ (see [GT01]). Moreover, one has

$$\nabla\phi = \frac{\delta}{q^2} \ \chi_{[q>\delta]} \ \nabla q,$$

where χ_A is the characteristic function, defined to be identically one on A and zero elsewhere. From Equation (1.3) and Assumption 1, we deduce:

$$\begin{aligned} h_0^3 \int_{x\in\Omega, \ q(x)>\delta} \frac{\Big|\nabla q\Big|^2}{q^2} \ \delta \ &\leq \ h_1 \int_{x\in\Omega, \ q(x)>\delta} \Big(H_\eta(p_1) - H_\eta(p_2)\Big) \ \frac{\partial q/\partial x_1}{q^2} \ \delta \\ &\leq \ \frac{h_1}{\eta} \int_{x\in\Omega, \ q(x)>\delta} \Big|\frac{\partial q/\partial x_1}{q}\Big| \ \delta. \end{aligned}$$

Then it follows:

$$\begin{aligned} h_0^3 \int_\Omega \Big|\nabla \ln\Big(1 + \frac{(q-\delta)^+}{\delta}\Big)\Big|^2 \ &\leq \ \frac{h_1}{\eta} \int_\Omega \Big|\frac{\partial}{\partial x_1} \ln\Big(1 + \frac{(q-\delta)^+}{\delta}\Big)\Big| \\ &\leq \ \frac{h_1}{\eta} \int_\Omega \Big|\nabla \ln\Big(1 + \frac{(q-\delta)^+}{\delta}\Big)\Big|. \end{aligned}$$

Applying Poincaré's inequality we obtain:

$$\int_\Omega \Big|\ln\Big(1 + \frac{(q-\delta)^+}{\delta}\Big)\Big|^2 \leq C,$$

where C depends on h_0, h_1, $|\Omega|$ and η but does not depend on δ. Then letting $\delta \to 0$,

$$q \leq 0, \quad \text{a.e.}$$

Exchanging the roles of p_1 and p_2 gives $q \geq 0$ a.e. so that, finally, $q = p_1 - p_2 = 0$ a.e. $\square$

1.1.3 Existence and uniqueness results for $(\mathcal{P}_\theta)$

Theorem 1.3. *Under Assumptions 1 and 2, problem $(\mathcal{P}_\theta)$ admits at least one solution.*

Proof. Existence of a solution is obtained by studying the behaviour of p_η when η goes to 0. First, let us notice that the following estimates hold:

$$\begin{aligned} \left\| H_\eta(p^\eta) \right\|_{L^\infty(\Omega)} &\leq C_1, \\ \left\| p^\eta \right\|_{H^1(\Omega)} &\leq C_2, \end{aligned}$$

where C_1 and C_2 do not depend on η. Indeed, they are easily obtained by considering the properties of H_η and using $p_\eta - p_a\ \psi$ as a test function. From the earlier estimates, one has:

(i) $\exists\ \theta \in L^\infty(\Omega)$, $H_\eta(p_\eta) \rightharpoonup \theta$, in $L^\infty(\Omega)$ weak-$\star$. In particular,

$$\int_\Omega H_\eta(p_\eta)\ h\ \frac{\partial \phi}{\partial x_1} \longrightarrow \int_\Omega \theta h\ \frac{\partial \phi}{\partial x_1}, \quad \forall\ \phi \in V_0,$$

(ii) $\exists\ p \in H^1(\Omega)$, $p_\eta \rightharpoonup p$, in $H^1(\Omega)$ and $p_\eta \to p$, in $L^2(\Omega)$. In particular,

$$\int_\Omega h^3\ \nabla p_\eta\ \nabla \phi \ \longrightarrow \int_\Omega h^3\ \nabla p\ \nabla \phi, \quad \forall\ \phi \in V_0.$$

From (i) and (ii), we deduce

$$\int_\Omega h^3\ \nabla p_\eta\ \nabla \phi = \int_\Omega H_\eta(p_\eta)\ h\ \frac{\partial \phi}{\partial x_1}, \quad \forall\ \phi \in V_0.$$

Moreover, considering Theorem III.9 of [Bre83], $p \in V_a$. It remains to prove the following properties to complete the proof of existence of a solution for the initial problem $(\mathcal{P}_\theta)$:

$$\begin{array}{rll} (i) & p \geq 0, & \text{a.e.}, \\ (ii) & 0 \leq \theta \leq 1, & \text{a.e.}, \\ (iii) & p\ (1-\theta) = 0, & \text{a.e.} \end{array}$$

■ Proof of (i) is deduced from positivity of p_η (see Lemma 1.2) and strong convergence of p_η to p in $L^2(\Omega)$.

■ Proof of (ii) is obtained considering the properties of the weak-$\star$ convergence (see Proposition III.12. in [Bre83]). Since we have

$$H_\eta(p_\eta) \rightharpoonup \theta, \quad \text{in } L^\infty(\Omega) \text{ weak-}\star,$$

then, $\Big\|\theta\Big\|_{L^\infty(\Omega)} \leq \lim\ \inf \Big\|H_\eta(p_\eta)\Big\|_{L^\infty(\Omega)} \leq 1$, and finally,

$$\theta \leq 1, \quad \text{a.e.}$$

Let us prove that $\theta \geq 0$ a.e. For this, we take $\chi_\eta = 1 - H_\eta(p_\eta)$ so that $\Big\|\chi_\eta\Big\|_{L^\infty(\Omega)} \leq 1$ and

$$\exists\ \chi \in L^\infty(\Omega),\ \chi_\eta \rightharpoonup \chi, \quad \text{in } L^\infty(\Omega) \text{ weak-}\star.$$

The weak-$\star$ topology is separated. Then $\chi = 1 - \theta$ and we have the following property:

$$\Big\|\chi\Big\|_{L^\infty(\Omega)} \leq \lim\ \inf \Big\|\chi_\eta\Big\|_{L^\infty(\Omega)} \leq 1,$$

which can be rewritten as

$$\Big\|1-\theta\Big\|_{L^\infty(\Omega)} \leq 1, \qquad \text{i.e.} \qquad \theta \geq 0, \quad \text{a.e.}$$

■ Proof of *(iii)* is obtained with the following method: let H denote the Heaviside graph. Since $p_\eta \geq 0$ (see Lemma 1.2), the following property holds:

$$\Big(1 - H(p_\eta)\Big)\ p_\eta = 0.$$

From this, we have $p_\eta\,(1 - H_\eta(p_\eta)) = p_\eta\,(H(p_\eta) - H_\eta(p_\eta))$. This term is analised in two steps:

- 1st step - Let ϕ be a function in $L^2(\Omega)$. Then,

$$\Big|\int_\Omega p_\eta\ (1{-}H_\eta(p_\eta))\ \phi{-}p\ (1{-}\theta)\ \phi\Big| = \Big|\int_\Omega (p_\eta{-}p)\ (1{-}H_\eta(p_\eta))\ \phi{+}\int_\Omega p\ (\theta{-}H_\eta(p_\eta))\ \phi\Big|.$$

  Using Cauchy-Schwarz inequality,

$$\Big|\int_\Omega p_\eta\ (1{-}H_\eta(p_\eta))\ \phi{-}p\ (1{-}\theta)\ \phi\Big| \leq \Big\|p_\eta{-}p\Big\|_{L^2(\Omega)}\Big\|\phi\Big\|_{L^2(\Omega)}{+}\Big|\int_\Omega p\ (\theta{-}H_\eta(p_\eta))\ \phi\Big|.$$

  With the L^2 strong convergence of p_η to p and the weak-$\star$ convergence of $1-H_\eta(p_\eta)$ to $1-\theta$, since $p\ \phi \in L^1(\Omega)$, we get

$$\Big|\int_\Omega p_\eta\ (1 - H_\eta(p_\eta))\ \phi - p\ (1-\theta)\ \phi\Big| \longrightarrow 0.$$

We have proved that

$$p_\eta\ (1 - H_\eta(p_\eta)) \rightharpoonup p\ (1-\theta), \quad \text{in } L^2(\Omega).$$

- 2nd step - Let ϕ be a function in $L^2(\Omega)$. Then, by construction of H_η,

$$\begin{aligned} \Big| \int_\Omega p_\eta \Big(H(p_\eta) - H_\eta(p_\eta)\Big)\ \phi \Big| &= \Big| \int_{\Omega^\eta} p_\eta \Big(1 - \frac{p_\eta}{\eta}\Big)\ \phi \Big| \\ &\le \Big| \int_{\Omega^\eta} \eta\ \phi \Big| \le \eta \int_\Omega \Big|\phi\Big|. \end{aligned}$$

with $\Omega^\eta = \{x \in \Omega, 0 \le p_\eta(x) \le \eta\}$. We have proved that

$$\Big(H(p_\eta) - H_\eta(p_\eta)\Big)\ p_\eta \rightharpoonup 0, \quad \text{in } L^2(\Omega).$$

From uniqueness of the weak limit in $L^2(\Omega)$ and the results stated in the two previous steps, we deduce:

$$p\ (1-\theta) = 0, \quad \text{in } L^2(\Omega).$$

□

We state a uniqueness result following an idea widely developped by Alvarez and Oujja in [AO03] for the unstationary case. The uniqueness result is based on a monotonicity result when comparing the value of two solutions on the upper boundary. Thus we first establish the following lemma:

Lemma 1.4. *Let (p_1, θ_1) and (p_2, θ_2) be two solutions of $(\mathcal{P}_\theta)$ with respective pressure boundary values p_a^1 and p_a^2 on Γ_a. Then,*

$$\int_\Omega h^3(x)\ \frac{\partial (p_1(x) - p_2(x))^+}{\partial x_2}\ \xi'(x_2)\ dx \le 0, \quad \forall\ \xi \in \mathcal{D}^+(0,1).$$

Proof.

■ **1st step: Test functions**

Let $X = (x_1, x_2)$ and $X' = (x'_1, x'_2)$ be two pairs of variables and let us define the following function:

$$\phi(X, X') = \xi\left(\frac{x_2 + x'_2}{2}\right)\ \rho_\varepsilon\left(\frac{x_2 - x'_2}{2}\right)\ \hat{\rho}_{\varepsilon'}\left(\frac{x_1 - x'_1}{2}\right),$$

where $\xi \in \mathcal{D}^+(0,1)$, $\rho_\varepsilon(r) = \dfrac{1}{\varepsilon}\ \rho\left(\dfrac{r}{\varepsilon}\right)$, $\hat{\rho}_{\varepsilon'}(r) = \dfrac{1}{\varepsilon'}\ \hat{\rho}\left(\dfrac{r}{\varepsilon'}\right)$. ρ and $\hat{\rho}$ are functions with supports in $(-1,1)$.

If $0 < \varepsilon < \mathrm{dist}(\mathrm{Supp}\xi, \partial[0,1])$, then the functions $\phi(X,\cdot)$ and $\phi(\cdot, X')$ vanish on the boundary $\Gamma_0 \cup \Gamma_a$ (see [Alv86] for the details and [AO03]). Moreover, in order to get a $2\pi x_1$ periodic function, we choose an even function $\hat{\rho}_{\varepsilon'}$ and redefine it when (x_1, x_1') belongs to the subset

$$T_{\varepsilon'} \cup S_{\varepsilon'} = \Big\{(x_1, x_1') \in [0, 2\pi] \times [0, 2\pi], \quad \Big|x_1 - x_1'\Big| \geq 2\pi - 2\varepsilon'\Big\},$$

by setting

$$\hat{\rho}_{\varepsilon'}\left(\frac{x_1 - x_1'}{2}\right) = \hat{\rho}_{\varepsilon'}\left(\frac{|x_1 - x_1'| - 2\pi}{2}\right).$$

Then we define the following function:

$$\eta_\nu(X, X') = \min\Big[\frac{(p_1(X) - p_2(X'))^+}{\nu}, \phi(X, X')\Big].$$

Thus, for fixed X' (resp. X), $\eta_\nu(\cdot, X')$ (resp. $\eta_\nu(X, \cdot)$ belongs to V_0.

■ **2nd step: Integral equality**

Let us denote Ω_1 and ∇_1 (resp. Ω_2 and ∇_2) the domain and the gradient vector for the variable X (resp. X'). For fixed X', let us use $\eta_\nu(\cdot, X')$ as a test function in the variational formulation of $(\mathcal{P}_\theta)$ with the variable X:

$$\int_{\Omega_1} h^3(X)\ \nabla_1\Big[p_1(X)\Big]\ \nabla_1\Big[\eta_\nu(X, X')\Big]\ dX = \int_{\Omega_1} \theta_1(X)\ h(X)\ \frac{\partial}{\partial x_1}\Big[\eta_\nu(X, X')\Big]\ dX.$$

Integrating the previous equation on Ω_2 gives us a first integral equality on $Q = \Omega_1 \times \Omega_2$. Applying the same method to the variable X' (and exchanging the roles of X and X'), we get a second integral equality. Then from periodicity and boundary conditions, it is possible to establish:

$$\begin{aligned}\sum_{i=1}^{2} \int_Q \Big[h^3(X)\ \nabla_i\Big(p_1(X)\Big) - h^3(X')\ \nabla_i\Big(p_2(X')\Big)\Big]\ \nabla_i\Big(\eta_\nu(X, X')\Big)\ dX\ dX' \\ = \sum_{i=1}^{2} \int_Q \Big(h(X) - h(X')\ \theta_2(X')\Big)\ \frac{\partial}{\partial x_i}\Big(\eta_\nu(X, X')\Big)\ dX\ dX'.\end{aligned}$$

■ **3rd step: Change of variables**

We make the following change of variables:

$$z = \frac{X + X'}{2}, \qquad \sigma = \frac{X - X'}{2}.$$

The integral equality becomes:

$$\int_{Q_{z,\sigma}} \Big[h^3(z+\sigma)\nabla_z\Big(p_1(z+\sigma)\Big) - h^3(z-\sigma)\nabla_z\Big(p_2(z-\sigma)\Big)\Big] \;\nabla_z\Big(\eta_\nu(z+\sigma,z-\sigma)\Big)\;dz\;d\sigma$$
$$= \int_{Q_{z,\sigma}} \Big(h(z+\sigma) - h(z-\sigma)\theta_2(z-\sigma)\Big)\;\frac{\partial}{\partial z_1}\Big(\eta_\nu(z+\sigma,z-\sigma)\Big)\;dz\;d\sigma,$$

where $Q_{z,\sigma}$ is the image of the domain Q through the change of variables. Let us consider the sets:

$$\begin{aligned} A_\nu &= \Big\{(z,\sigma)\in Q_{z,\sigma},\ \frac{(p_1(z+\sigma)-p_2(z-\sigma))^+}{\nu} > \phi(z+\sigma,z-\sigma)\Big\},\\ B_\nu &= \Big\{(z,\sigma)\in Q_{z,\sigma},\ \frac{(p_1(z+\sigma)-p_2(z-\sigma))^+}{\nu} \le \phi(z+\sigma,z-\sigma)\Big\}. \end{aligned}$$

Let us denote I_1 (resp. I_2) the contribution of A_ν (resp. B_ν) in the first integral and let us denote J_1 (resp. J_2) the contribution of A_ν (resp. B_ν) in the second integral. Then we have: $I_1 + I_2 = J_1 + J_2$.

■ **4th step: Study of the integrals**

• Let us study J_1: since ϕ does not depend on z_1, one gets: $J_1 = 0$.

• Let us study J_2: let us recall the expression of J_2:

$$\begin{aligned} J_2 &= \int_{B_\nu} \Big(h(z+\sigma) - h(z-\sigma)\;\theta_2(z-\sigma)\Big)\;\frac{\partial}{\partial z_1}\Big(\frac{(p_1(z+\sigma)-p_2(z-\sigma))^+}{\nu}\Big)\;dz\;d\sigma\\ &= \int_{B_\nu} \Big(h(z+\sigma) - h(z-\sigma)\Big)\;\frac{\partial}{\partial z_1}\Big(\frac{(p_1(z+\sigma)-p_2(z-\sigma))^+}{\nu}\Big)\;dz\;d\sigma\\ &\quad+\int_{B_\nu} h(z-\sigma)\;\Big(1-\theta_2(z-\sigma)\Big)\;\frac{\partial}{\partial z_1}\Big(\frac{(p_1(z+\sigma)-p_2(z-\sigma))^+}{\nu}\Big)\;dz\;d\sigma. \end{aligned}$$

The first integral J_2^1 can be rewritten as

$$\int_{Q_{z,\sigma}} \Big(h(z+\sigma)-h(z-\sigma)\Big)\;\frac{\partial}{\partial z_1}\Big(\min\Big[\frac{(p_1(z+\sigma)-p_2(z-\sigma))^+}{\nu},\phi(z+\sigma,z-\sigma)\Big]\Big)\;dz\;d\sigma.$$

Integrating by parts, letting $\nu\to 0$, and using Lebesgue theorem, we get:

$$\lim_{\nu\to 0} J_2^1 = \int_{Q^{\triangle}_{z,\sigma}} \Big[\frac{\partial h}{\partial z_1}(z+\sigma) - \frac{\partial h}{\partial z_1}(z-\sigma)\Big]\;\xi(z_2)\;\rho_\varepsilon(\sigma_2)\;\hat\rho_{\varepsilon'}(\sigma_1)\;dz\;d\sigma.$$

with $Q^{\triangle}_{z,\sigma} = \{(z,\sigma)\in Q_{z,\sigma},\ p_1(z+\sigma) > p_2(z-\sigma)\}$. Moreover

(i) $\mathrm{Supp}(\rho_\varepsilon)\subset[-\varepsilon,\varepsilon]$, $\mathrm{Supp}(\hat\rho_{\varepsilon'})\subset[-\varepsilon',\varepsilon']$,

(ii) $\dfrac{\partial h}{\partial z_1}$ is a Lipschitz continuous function, so that we get

$$\Big| \lim_{\nu\to 0} J_2^1 \Big| \le C(\varepsilon+\varepsilon') \int_{Q_{z,\sigma}} \xi(z_2)\ \rho_\varepsilon(\sigma_2)\ \hat\rho_{\varepsilon'}(\sigma_1)\ dz\ d\sigma,$$

and, finally,

$$\lim_{\varepsilon,\varepsilon'\to 0} \Big| \lim_{\nu\to 0} J_2^1 \Big| = 0.$$

The second integral can be rewritten in the old variables as

$$\begin{aligned} J_2^2 &= \int_Q h(X')\ \Big(1-\theta_2(X')\Big)\ \Big[\Big(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_1'}\Big)\Big(\frac{p_1(X)-p_2(X')}{\nu}\Big)\Big]\ dX\ dX' \\ &= \int_Q h(X')\ \Big(1-\theta_2(X')\Big)\frac{\partial}{\partial x_1}\Big(\frac{p_1(X)}{\nu}\Big)\ dX\ dX' \end{aligned}$$

since $1-\theta_2 = 0$ when $p_2 > 0$. Rewriting the integral, one gets:

$$\begin{aligned} J_2^2 &= \int_Q h(X')\ \Big(1-\theta_2(X')\Big)\ \frac{\partial}{\partial x_1}\min\Big[\frac{p_1(X)}{\nu},\phi(X,X')\Big]\ dX\ dX' \\ &\qquad - \int_{A_\nu} h(X')\ \Big(1-\theta_2(X')\Big)\ \frac{\partial}{\partial x_1}\Big(\phi(X,X')\Big)\ dX\ dX' \\ &= \int_{B_\nu} h(X')\ \Big(1-\theta_2(X')\Big)\ \frac{\partial}{\partial x_1}\Big(\phi(X,X')\Big)\ dX\ dX', \end{aligned}$$

using the Green formula with periodicity and boundary conditions. Since the function

$$h(X')\ \Big(1-\theta_2(X')\Big)\ \frac{\partial}{\partial x_1}\phi(X,X')$$

is bounded for each ε, ε', we conclude

$$\Big| \lim_{\nu\to 0} J_2^2 \Big| \le \lim_{\nu\to 0} C\Big|B_\nu\Big| = 0,$$

and finally,

$$\lim_{\varepsilon,\varepsilon'\to 0} \Big| \lim_{\nu\to 0} J_2^2 \Big| = 0.$$

• Let us study I_1, which is given by the following expression

$$\int_{A_\nu} \Big[h^3(z+\sigma)\nabla_z p_1(z+\sigma) - h^3(z-\sigma)\nabla_z p_2(z-\sigma)\Big]\ \nabla_z\Big(\xi(z_2)\ \rho_\varepsilon(\sigma_2)\ \hat\rho_{\varepsilon'}(\sigma_1)\Big)\ dz\ d\sigma.$$

By Lebesgue theorem, $\lim\limits_{\nu\to 0} I_1$ is equal to

$$\begin{aligned}
&\int_{Q^{\triangle}_{z,t}} \Big[h^3(z+\sigma)\frac{\partial p_1(z+\sigma)}{\partial z_2} - h^3(z-\sigma)\frac{\partial p_2(z-\sigma)}{\partial z_2}\Big]\ \xi'(z_2)\ \rho_\varepsilon(\sigma_2)\ \hat{\rho}_{\varepsilon'}(\sigma_1)\ dz\ d\sigma \\
&\qquad = \int_{Q^{\triangle}_{z,t}} \Big[h^3(z+\sigma)\ - h^3(z-\sigma)\Big]\ \frac{\partial p_2(z-\sigma)}{\partial z_2}\ \xi'(z_2)\ \rho_\varepsilon(\sigma_2)\ \hat{\rho}_{\varepsilon'}(\sigma_1)\ dz\ d\sigma \\
&\qquad\quad + \int_{Q^{\triangle}_{z,t}} h^3(z+\sigma)\ \frac{\partial p_1(z+\sigma) - p_2(z-\sigma)}{\partial z_2}\ \xi'(z_2)\ \rho_\varepsilon(\sigma_2)\ \hat{\rho}_{\varepsilon'}(\sigma_1)\ dz\ d\sigma,
\end{aligned}$$

with, again, $Q^{\triangle}_{z,\sigma} = \{(z,\sigma) \in Q_{z,\sigma},\ p_1(z+\sigma) > p_2(z-\sigma)\}$. Using the properties of ρ_ε, $\hat{\rho}_{\varepsilon'}$ and since h^3 is a Lipschitz continuous function, it is easy to conclude that the first integral goes to 0 when $\varepsilon,\ \varepsilon' \to 0$. Then we obtain, studying the behaviour of the second integral (see [Alv86] for the details):

$$\lim_{\nu,\varepsilon,\varepsilon'\to 0} I_1 = \int_\Omega h^3(x)\ \frac{\partial\Big(p_1(x) - p_2(x)\Big)^+}{\partial x_2}\ \xi'(x_2)\ dx.$$

• Let us study I_2:
Rewriting I_2 in the old variables gives:

$$\begin{aligned}
I_2 \quad = \quad & \int_{B_\nu} \Big[h^3(X)\ \Big|\nabla_1 \frac{p_1(X)}{\nu}\Big|^2 + h^3(X')\ \Big|\nabla_2 \frac{p_2(X')}{\nu}\Big|^2\Big]\ dX\ dX' \\
& - \int_{B_\nu} h^3(X)\ \nabla_1 p_1(X)\ \nabla_2\Big(\frac{p_2(X')}{\nu}\Big)\ dX\ dX' \\
& - \int_{B_\nu} h^3(X')\ \nabla_2 p_2(X')\ \nabla_1\Big(\frac{p_1(X)}{\nu}\Big)\ dX\ dX'.
\end{aligned}$$

The first integral is positive. The second integral satisfies:

$$\begin{aligned}
&\int_{B_\nu} h^3(X)\ \nabla_1 p_1(X)\ \nabla_2\Big(\frac{p_2(X')}{\nu}\Big)\ dX\ dX' \\
&\qquad = -\int_{B_\nu} h^3(X)\ \nabla_1 p_1(X)\ \nabla_2\Big(\frac{p_1(X) - p_2(X')}{\nu}\Big)\ dX\ dX' \\
&\qquad = -\int_{B_\nu} h^3(X)\ \nabla_1 p_1(X)\ \nabla_2\Big(\phi(X,X')\Big)\ dX\ dX'.
\end{aligned}$$

By Hölder inequality and since $\lim\limits_{\nu\to 0} |B_\nu| = 0$, one gets:

$$\begin{aligned}
&\lim_{\nu\to 0} \int_{B_\nu} h^3(X)\ \nabla_1 p_1(X)\ \nabla_2\Big(\frac{p_2(X')}{\nu}\Big)\ dX\ dX' \\
&\qquad \leq \lim_{\nu\to 0} |B_\nu|^{1/2} \Big(\int_Q h^6(X)\ \Big|\nabla_1 p_1(X)\Big|^2\ \Big|\nabla_2\phi(X,X')\Big|^2\ dX\ dX'\Big)^{1/2} = 0.
\end{aligned}$$

In a similar way,

$$\lim_{\nu\to 0}\int_{B_\nu} h^3(X')\ \nabla_2 p_2(X')\ \nabla_1\Big(\frac{p_1(X)}{\nu}\Big)\ dX\ dX' = 0,$$

and we deduce

$$\lim_{\nu,\varepsilon,\varepsilon'\to 0} I_2 \geq 0.$$

Now passing to the limit $(\nu, \varepsilon, \varepsilon' \to 0)$ in the integral equality concludes the proof. □

Theorem 1.5. *Let (p_1, θ_1) and (p_2, θ_2) two solutions of $(\mathcal{P}_\theta)$ with respective pressure boundary values p_a^1 and p_a^2 on Γ_a. Let us suppose that $p_a^1 \leq p_a^2$. Then*

$$p_1 \leq p_2, \quad \text{a.e.}$$

Proof. From Lemma 1.4, denoting $f = (p_1 - p_2)^+$, we have, for every $\xi \in \mathcal{D}^+(0,1)$,

$$\int_\Omega h^3(x)\ \frac{\partial f}{\partial x_2}(x)\ \xi'(x_2)\ dx \leq 0.$$

Then one gets:

$$\int_\Omega f(x)\ h^3(x)\ \xi''(x_2)\ dx + \int_\Omega f(x)\ \frac{\partial h^3}{\partial x_2}(x)\ \xi'(x_2)\ dx \geq 0, \quad \forall\ \xi \in \mathcal{D}^+(0,1).$$

Using the following notations:

$$a(x_2) = \int_0^{2\pi} f(x)\ h^3(x)\ dx_1, \qquad b(x_2) = \int_0^{2\pi} f(x)\frac{\partial h^3}{\partial x_2}(x)\ dx_1,$$

we get:

$$\int_0^1 a(x_2)\ \xi''(x_2)\ dx_2 + \int_0^1 b(x_2)\ \xi'(x_2)\ dx_2 \geq 0,\ \forall\ \xi \in \mathcal{D}^+(0,1). \tag{1.4}$$

Now let us suppose that $a(x_2) > 0$, $\forall\ x_2 \in (y_0, y_1) \subset (0,1)$ and let ξ_0 be a solution of the two points boundary problem:

$$a(x_2)\ \xi''(x_2) + b(x_2)\ \xi'(x_2) = a(x_2)\ \psi''(x_2), \qquad \xi(y_0) = \xi(y_1) = 0, \tag{1.5}$$

where $\psi \in C^\infty[y_0, y_1]$ satisfying $\psi''(x_2) < 0$, $\forall\ x_2 \in [y_0, y_1]$. From the minimum principle, $\xi_0(x_2) \geq 0$, $\forall\ x_2 \in [y_0, y_1]$. Then we define a regularizing function g on $[y_0, y_1]$ such that $g\ \xi_0$ is a test function for Equation (1.4) and $g = 1$ on $[y_0 + \delta, y_1 - \delta]$. More precisely, let

δ be a positive parameter and g the function defined on $[y_0, y_1]$ by

$$g(x_2) = \begin{cases} 2\left(\dfrac{x_2 - y_0{}^2}{\delta}\right), & x_2 \in (y_0, y_0 + \delta/2) \\ 1 - 2\left(1 - \dfrac{x_2 - y_0{}^2}{\delta}\right), & x_2 \in (y_0 + \delta/2, y_0 + \delta) \\ 1, & x_2 \in (y_0 + \delta, y_1 - \delta) \\ 1 - 2\left(1 - \dfrac{y_1 - x_2{}^2}{\delta}\right), & x_2 \in (y_1 - \delta, y_1 - \delta/2) \\ 2\left(\dfrac{y_1 - x_2{}^2}{\delta}\right), & x_2 \in (y_1 - \delta/2, y_1) \end{cases}$$

This function satisfies $g(y_0) = g(y_1) = 0$ and $g'(y_0) = g'(y_1) = 0$. Define $\widetilde{\xi}(x_2) = g(x_2)\ \xi_0(x_2)$, $\forall x_2 \in [y_0, y_1]$. We have $\widetilde{\xi} \in C^2(y_0, y_1)$, $\widetilde{\xi}(y_0) = \widetilde{\xi}(y_1) = 0$ and $\widetilde{\xi}'(y_0) = \widetilde{\xi}'(y_1) = 0$. Therefore, we can take $\xi = \widetilde{\xi}$ in Equation (1.4) and get

$$\int_{y_0}^{y_1} a(x_2)\ \widetilde{\xi}''(x_2) + b(x_2)\ \widetilde{\xi}'(x_2)\ dx_2 \geq 0.$$

By separating the integration intervals, we decompose this integral in the form

$$\begin{aligned} &\int_{y_0}^{y_0+\delta} a\ (g\ \xi_0)'' + b\ (g\ \xi_0)' \ + \int_{y_0+\delta}^{y_1-\delta} a\ \xi_0'' + b\ \xi_0' \\ &\qquad + \int_{y_1-\delta}^{y_1} a\ (g\ \xi_0)'' + b\ (g\ \xi_0)' \geq 0. \end{aligned} \tag{1.6}$$

From Equation (1.5), the second integral is strictly negative, and for the two other integrals:

$$\begin{aligned} \int_{y_0}^{y_0+\delta} a\ (g\ \xi_0)'' + b\ (g\ \xi_0)' &= \int_{y_0}^{y_0+\delta} a\ (g''\ \xi_0 + 2g'\ \xi_0 + g\ \xi_0'') + b\ (g'\ \xi_0 + g\ \xi_0') \\ &= \int_{y_0}^{y_0+\delta} \big(a\ g''\ \xi_0 + 2a\ g'\ \xi_0 + a\ g\ \xi_0'' + (b\ g'\ \xi_0 + b\ g\ \xi_0'\big). \end{aligned} \tag{1.7}$$

Since $\mid g' \mid \approx 1/\delta$, $\mid g'' \mid \approx 1/\delta^2$ and being the functions a and ξ_0 continuous in the interval $(y_0, y_0 + \delta)$, the terms under the last integral in Equation (1.7) are bounded and we obtain

$$\int_{y_0}^{y_0+\delta} a\ (g\ \xi_0)'' + b\ (g\ \xi_0)' \approx \delta.$$

In the same way, we have

$$\int_{y_1-\delta}^{y_1} a\ (g\ \xi_0)'' + b\ (g\ \xi_0)' \approx \delta.$$

Passing to the limit ($\delta \to 0$) in Inequality (1.6), one gets:

$$\int_{y_0}^{y_1} a\ \xi_0'' + b\ \xi_0' \geq 0.$$

But we have also:

$$\int_{y_0}^{y_1} a\ \xi_0'' + b\ \xi_0' = \int_{y_0}^{y_1} a\ \psi'' < 0.$$

Thus, $a(x_2) = \displaystyle\int_0^{2\pi} f(x)\ h^3(x)\ dx_1 \leq 0$ on $(0,1)$, i.e.

$$\int_0^{2\pi} (p_1(x) - p_2(x))^+\ h^3(x)\ dx_1 \leq 0,$$

and we conclude $p_1 \leq p_2$ a.e. □

Theorem 1.6. *Under Assumptions 1 and 2, problem $(\mathcal{P}_\theta)$ admits at least one solution (p, θ) whose pressure p is unique. Moreover, if there exists a set of positive measure where $p(x_1, x_2) > 0$, for any $x_2 > 0$, then the saturation θ is unique.*

Proof.
■ Uniqueness of the pressure is obtained from Theorem 1.5.
■ Let us consider (p, θ_1) and (p, θ_2) two solutions. Then we get, by means of substraction:

$$\int_\Omega h\ (\theta_1 - \theta_2)\ \frac{\partial \psi}{\partial x_1} = 0, \quad \forall\ \psi \in V_0, \quad \text{and} \quad \frac{\partial h\ (\theta_1 - \theta_2)}{\partial x_1} = 0, \quad \text{in } \mathcal{D}'(\Omega),$$

so that $h\ (\theta_1 - \theta_2)$ is a function only depending on the x_2 variable, almost everywhere in Ω. In particular, if there exists a set of positive measure where $\theta_1(x) = \theta_2(x)$, for every $x_2 > 0$, then $\theta_1 = \theta_2$ a.e. □

We give a supplementary result:

Corollary 1.7. *Under Assumptions 1 and 2 and if h can be written under the form $h(x_1, x_2) = h_1(x_1)h_2(x_2)$, then problem $(\mathcal{P}_\theta)$ admits a unique solution.*

Proof. By Theorem 1.6, it is sufficient to prove that, for any $x_2 > 0$, there exists a set of positive measure, where $p(x_1, x_2) > 0$. Let ψ be a test function only depending on x_2. Then we have

$$\int_\Omega h^3 \frac{\partial p}{\partial x_2} \psi' = 0, \text{ i.e. } \int_0^1 \left(\int_0^{2\pi} h^3(x) \frac{\partial p}{\partial x_2}(x)\ dx_1 \right) \psi'(x_2)\ dx_2 = 0.$$

Thus, we get

$$\int_0^{2\pi} h^3(x) \frac{\partial p}{\partial x_2}(x)\ dx_1 = C,$$

where C is a real constant. Since h can be written under the form $h(x_1, x_2) = h_1(x_1)h_2(x_2)$, dividing the previous equality by $h_2^3(x_2)$ gives

$$\frac{\partial}{\partial x_2}\left(\int_0^{2\pi} h_1^3(x_1)\ p(x)\ dx_1\right) = \frac{C}{h_2^3(x_2)}.$$

Integrating the previous equality and taking into account the boundary conditions on the pressure,

$$\left(\int_0^{2\pi} h_1^3(x_1)\ p(x)\ dx_1\right) = \frac{\displaystyle\int_0^{2\pi} h_1^3}{\displaystyle\int_0^1 h_2^{-3}}\ p_a \int_0^{x_2} h_2^{-3}(t)\ dt \quad > 0, \quad \forall\ x_2 > 0. \tag{1.8}$$

We deduce from Equation (1.8) that, for any $x_2 > 0$, there exists a set of positive measure, where $p(x) > 0$. $\square$

The next sections deal with homogenization of the lubrication problem, using two-scale convergence techniques which have been introduced by Nguetseng in [Ngu89], and further developped by Allaire [All92], Cioranescu, Damlamian and Griso [CDG02] and Lukkassen, Nguetseng and Wall [LNW02].

1.2 Homogenization of the lubrication problem

In the whole section, $\Omega =]0, 2\pi[\times]0, 1[$ and $Y =]0, 1[\times]0, 1[$. Now we introduce the roughness of the upper surface; the roughness is supposed to be periodic, characterized by a small parameter ε denoting the roughness spacing. Due to the shape of the Reynolds equation, oscillating data appear in both sides of the equation. So we are led to consider the following problem $(\mathcal{P}_\theta^\varepsilon)$ and assumptions:

Assumption 3. *Let a and b be functions such that:*

(i) $a \in L^2_\sharp(\Omega; C_\sharp(Y))$ or $a \in L^2_\sharp(Y; C_\sharp(\overline{\Omega}))$,

(ii) $b \in L^2_\sharp(\Omega; C_\sharp(Y))$ or $b \in L^2_\sharp(Y; C_\sharp(\overline{\Omega}))$,

(iii) $\exists\ m_a, M_a, \quad \forall(x, y) \in \Omega \times Y, \quad 0 < m_a \leq a(x, y) \leq M_a,$

(iv) $\exists\ m_b, M_b, \quad \forall(x, y) \in \Omega \times Y, \quad 0 < m_b \leq b(x, y) \leq M_b.$

We introduce the following functions defined on Ω:

$$a_\varepsilon(x) = a\left(x, \frac{x}{\varepsilon}\right), \quad b_\varepsilon(x) = b\left(x, \frac{x}{\varepsilon}\right).$$

Then we introduce the following problem:

$$(\mathcal{P}_\theta^\varepsilon)\begin{cases} \textit{Find } (p_\varepsilon,\theta_\varepsilon)\in V_a\times L^\infty(\Omega) \textit{ such that:}\\ \displaystyle\int_\Omega a_\varepsilon\ \nabla p_\varepsilon\ \nabla\phi=\int_\Omega \theta_\varepsilon\ b_\varepsilon\ \frac{\partial\phi}{\partial x_1},\quad \forall\ \phi\in V_0,\\ p_\varepsilon\geq 0,\quad p_\varepsilon\ (1-\theta_\varepsilon)=0,\quad 0\leq\theta_\varepsilon\leq 1,\quad a.e. \end{cases}$$

Existence and uniqueness results have been discussed in Section 1.1. Our purpose is to discuss the behaviour of problem $(\mathcal{P}_\theta^\varepsilon)$ when ε goes to 0, using two-scale convergence techniques.

1.2.1 Preliminaries to the two-scale convergence technique

First we recall some useful definitions and results for the two-scale convergence (see [All92, CDG02, LNW02]).

Lemma 1.8. *The separable Banach space $L^2(\Omega;C_\sharp(Y))$ is dense in $L^2(\Omega\times Y)$. Moreover, if $f\in L^2(\Omega;C_\sharp(Y))$, then $x\mapsto\sigma_\varepsilon(f)(x)=f(x,x/\varepsilon)$ is a measurable function such that*

$$\Big\|\sigma_\varepsilon(f)\Big\|_{L^2(\Omega)}\leq\Big\|f\Big\|_{L^2(\Omega;C_\sharp(Y))}$$

Definition 1. *The sequence $u_\varepsilon\in L^2(\Omega)$ two-scale converges to a limit $u_0\in L^2(\Omega\times Y)$ if, for any $\psi\in L^2(\Omega;C_\sharp(Y))$, one has*

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\ \psi\left(x,\frac{x}{\varepsilon}\right)\ dx=\int_\Omega\int_Y u_0(x,y)\ \psi(x,y)\ dy\ dx.$$

Lemma 1.9. *Let u_ε be a bounded sequence in $L^2(\Omega)$. Then there exists $u_0\in L^2(\Omega\times Y)$ such that, up to a subsequence, u_ε two-scale converges to u_0.*

Lemma 1.10. *Let u_ε be a bounded sequence in $H^1(\Omega)$, which weakly converges to a limit $u_0\in H^1(\Omega)$. Then u_ε two-scale converges to u_0 and there exists a function $u_1\in L^2(\Omega;H^1_\sharp(Y)/\mathbb{R})$ such that, up to a subsequence, ∇u_ε two-scale converges to $\nabla u_0+\nabla_y u_1$.*

1.2.2 Two-scale convergence results

In this subsection, $(p_\varepsilon,\theta_\varepsilon)$ denotes a solution of problem $(\mathcal{P}_\theta^\varepsilon)$.

Lemma 1.11. *There exists $p_0\in V_a$ such that, up to a subsequence:*

$$p_\varepsilon\rightharpoonup p_0\ \text{ in } H^1(\Omega)\qquad\text{and}\qquad p_\varepsilon\to p_0\ \text{ in } L^2(\Omega).$$

We have also the following two-scale convergences:

(i) p_ε *two-scale converges to* p_0. *Moreover, there exists* $p_1 \in L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$ *and a subsequence* ε', *still denoted* ε, *such that* ∇p_ε *two-scale converges to* $\nabla p_0 + \nabla_y p_1$.

(ii) There exists $\theta_0 \in L^2(\Omega \times Y)$ *and a subsequence* ε", *still denoted* ε, *such that* θ_ε *two-scale converges to* θ_0.

Moreover, $p_0 \geq 0$ *a.e.*

Proof. Since $0 \leq \theta_\varepsilon \leq 1$, θ_ε is bounded in $L^\infty(\Omega)$ and in $L^2(\Omega)$, so that $\|\theta_\varepsilon\|_{L^2(\Omega)} \leq C_1$, where C_1 only depends on Ω. Moreover, from Assumptions 3 (ii)–(iv), properties of θ_ε and the Cauchy-Schwarz inequality, we get the estimates on p_ε by using $p_\varepsilon - \overline{p}_a$ (with $\overline{p}_a$ a regular function such that $p_\varepsilon - \overline{p}_a \in V_0$) as a test function and Poincaré-Friedrichs inequality so that $\|p_\varepsilon\|_{H^1(\Omega)} \leq C_2$ where C_2 only depends on Ω. The convergence results are the consequence of the previous estimates (see Lemmas 1.9 and 1.10, or Proposition 1.14 in [All92], Theorem 13 in [LNW02]). Finally $p_0 \geq 0$ a.e. due to the properties of p_ε, by passing to the limit. □

Now, we give the properties of the two-scale limits p_0 and θ_0, which are quite similar to the ones of the initial functions p_ε and θ_ε. These properties are obtained by means of two-scale convergence techniques.

Proposition 1.12. $0 \leq \theta_0 \leq 1$ *a.e.*

Proof. Let us introduce the classical notation $w^+ = \max(w, 0)$ and $w^- = -\min(w, 0)$, for any $w \in L^2(\Omega \times Y)$. Since $L^2(\Omega; C_\sharp(Y))$ is dense in $L^2(\Omega \times Y)$ (see Theorem 3 in [LNW02]), let us consider a sequence $\phi_n \in L^2(\Omega; C_\sharp(Y))$, $\phi_n \geq 0$, which strongly converges to θ_0^- in

$L^2(\Omega \times Y)$ (note that such a sequence exists[2]). Thus, defining the following sequences

$$A_n^\varepsilon = \int_\Omega \theta_\varepsilon(x)\ \phi_n\left(x, \frac{x}{\varepsilon}\right)\ dx, \quad A_n^\star = \int_{\Omega\times Y} \theta_0(x,y)\ \phi_n\left(x,y\right)\ dy\ dx,$$

we have, using the two-scale convergence of θ_ε,

$$\lim_{\varepsilon\to 0} A_n^\varepsilon = A_n^\star.$$

Obviously, A_n^ε is a sequence of positive numbers so that we have also: $A_n^\star \geq 0$. Now letting $n \to +\infty$, we have:

$$\lim_{n\to+\infty} A_n^\star = -\int_{\Omega\times Y} (\theta_0^-)^2 = A \quad (\leq 0).$$

Thus, $A_n^\star$ being a sequence of positive numbers, $A \geq 0$ so that, finally, $A = 0$. Thus, we have proved that $\theta_0^- = 0$ a.e. Similarly, it can be proved that $(1-\theta_0)^- = 0$ a.e. □

Proposition 1.13. *$p_0\ (1-\theta_0) = 0$ a.e.*

Proof. By uniqueness of the two-scale limit (see [All92, LNW02]), it is sufficient to prove that $p_\varepsilon\ (1-\theta_\varepsilon)$ two-scale converges to $p_0\ (1-\theta_0)$. As p_ε two-scale converges to p_0, let us prove that $p_\varepsilon\ \theta_\varepsilon$ two-scale converges to $p_0\ \theta_0$. The sequence $\{\theta_\varepsilon p_\varepsilon\}$ is bounded in $L^2(\Omega)$. Consequently, it remains to prove (see Proposition 1 in [LNW02]):

$$\int_\Omega p_\varepsilon(x)\ \theta_\varepsilon(x)\ \phi\left(x, \frac{x}{\varepsilon}\right)\ dx \to \int_{\Omega\times Y} p_0(x)\ \theta_0(x,y)\ \phi(x,y)\ dy\ dx,$$

[2]Let $\psi \in L^2(\Omega\times Y)$, $\psi \geq 0$. By Theorem 3 in [LNW02], there exists a sequence $\psi_n \in L^2(\Omega; C_\sharp(Y))$ such that ψ_n strongly converges to ψ in $L^2(\Omega\times Y)$. Now it is sufficient to prove that

(i) $\psi_n^+ \in L^2(\Omega; C_\sharp(Y))$,

(ii) ψ_n^+ strongly converges to ψ in $L^2(\Omega \times Y)$ up to a subsequence.

We have the following characterization of $L^2(\Omega; C_\sharp(Y))$ (see Theorem 1 of [LNW02]): a function f belongs to $L^2(\Omega\times Y)$ if and only if there exists a subset E of measure zero in Ω such that:

(a) for any $x \in \Omega \setminus E$, the function $y \to f(x,y)$ is continuous and Y periodic,

(b) for any $y \in Y$, the function $x \to f(x,y)$ is measurable,

(c) the function $x \to \sup_{y\in Y} |f(x,y)|$ has finite $L^2(\Omega)$ norm.

Thus, it is obvious that if $\psi_n \in L^2(\Omega; C_\sharp(Y))$, then $\psi_n^+ \in L^2(\Omega; C_\sharp(Y))$. It remains to prove that, up to a subsequence, ψ_n^- strongly converges to 0 in $L^2(\Omega\times Y)$. Thus, by Theorem IV.9 in [Bre83], as $\psi_n,\ \psi \in L^2(\Omega\times Y)$ with $\|\psi_n - \psi\|_{L^2(\Omega\times Y)} \to 0$, there exists a subsequence ψ_{n_k} such that

(a) $\psi_{n_k} \to \psi$ a.e.,

(b) $|\psi_{n_k}(x,y)| \leq \Lambda(x,y)$, for all n_k, a.e., with $h \in L^2(\Omega\times Y)$.

Now, since $\psi_{n_k}^- \to 0$ a.e. on $\Omega\times Y$ and $|\psi_{n_k}^-(x,y)| \leq |\psi_{n_k}(x,y)| \leq \Lambda(x,y)$ we state from the Lebesgue theorem that $\|\psi_{n_k}^-\|_{L^2(\Omega\times Y)} \to 0$, and the proof is concluded.

for all $\phi \in D(\Omega; C^\infty_\sharp(Y))$. Let ϕ be a function in $D(\Omega; C^\infty_\sharp(Y))$ and let α_ε be defined by:

$$\alpha_\varepsilon = \int_\Omega p_\varepsilon(x)\ \theta_\varepsilon(x)\ \phi\left(x, \frac{x}{\varepsilon}\right)\ dx - \int_{\Omega\times Y} p_0(x)\ \theta_0(x,y)\ \phi(x,y)\ dy\ dx.$$

Our purpose is to prove that α_ε tends to 0. Then we have:

$$\begin{aligned}\alpha_\varepsilon \quad &= \quad \underbrace{\int_\Omega \Big[p_\varepsilon(x) - p_0(x)\Big]\ \theta_\varepsilon(x)\ \phi\left(x, \frac{x}{\varepsilon}\right)\ dx}_{\Lambda^1_\varepsilon} \\ &\quad + \underbrace{\int_\Omega p_0(x)\ \theta_\varepsilon(x)\ \phi(x, \frac{x}{\varepsilon})\ dx - \int_{\Omega\times Y} p_0(x)\ \theta_0(x,y)\ \phi(x,y)\ dy\ dx}_{\Lambda^2_\varepsilon}.\end{aligned}$$

■ Using the Cauchy-Schwarz inequality and Lemma 1.8 (see also Lemma 1.3 in [All92] or Theorem 3 in [LNW02]), we have:

$$\left|\Lambda^1_\varepsilon\right| \leq \ \left\|p_\varepsilon - p_0\right\|_{L^2(\Omega)} \left\|\sigma_\varepsilon(\phi)\right\|_{L^2(\Omega)} \leq \ \left\|p_\varepsilon - p_0\right\|_{L^2(\Omega)} \left\|\phi\right\|_{L^2(\Omega; C_\sharp(Y))}.$$

As $p_\varepsilon \longrightarrow p_0$ in $L^2(\Omega)$, we have: $\left|\Lambda^1_\varepsilon\right| \longrightarrow 0$.

■ In order to prove that $\Lambda^2_\varepsilon \to 0$, since θ_ε two-scale converges to θ_0, it is sufficient to prove that $(x,y) \to \psi(x,y) = p_0(x)\ \phi(x,y)$ is an admissible test function for the two-scale convergence (i.e. $\psi \in L^2(\Omega; C_\sharp(Y))$).

Let us prove that $(x,y) \to p_0(x)\ \phi(x,y) \in L^2(\Omega; C_\sharp(Y))$ for every $\phi \in D(\Omega; C^\infty_\sharp(Y))$.

▷ With $\phi \in D(\Omega; C^\infty_\sharp(Y))$ and $p_0 \in H^1(\Omega)$, we have for a.e. x in Ω:

$$p_0(x)\ \phi(x,\cdot) \in C^\infty_\sharp(Y) \subset C_\sharp(Y).$$

▷ Let us denote $\Psi_0(x,y) = p_0(x)\,\phi(x,y)$. As $p_0 \in H^1(\Omega) \subset L^4(\Omega)$, $\phi \in D(\Omega; C^\infty_\sharp(Y)) \subset L^4(\Omega; C_\sharp(Y))$, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\left\|\Psi_0\right\|^2_{L^2(\Omega; C_\sharp(Y))} \quad &= \quad \int_\Omega p_0^2(x)\ \sup_{y\in Y} \left|\phi(x,y)\right|^2\ dx \\ &\leq \quad \Big(\int_\Omega\ p_0^4(x)\ dx\Big)^{1/2}\ \Big(\int_\Omega \sup_{y\in Y} \left|\phi(x,y)\right|^4\ dx\Big)^{1/2} < +\infty.\end{aligned}$$

We have proved that $(x,y) \to p_0(x)\ \phi(x,y) \in L^2(\Omega; C_\sharp(Y))$ for any function $\phi \in$

$D(\Omega; C^\infty_\sharp(Y))$. Then, $\Lambda^2_\varepsilon \to 0$.

□

1.2.3 Homogenization of the lubrication problem (general case)

Using an idea developped in [All92], one has the following macro-microscopic decomposition:

Theorem 1.14. *From the initial formulation,*

- *Macroscopic equation:*

$$\int_\Omega \Big(\int_Y a\Big[\nabla p_0 + \nabla_y p_1\Big]\Big) \nabla\phi = \int_\Omega \Big(\int_Y \theta_0 \; b\Big) \; \frac{\partial \phi}{\partial x_1}, \tag{1.9}$$

for every ϕ in V_0.

- *Microscopic equation:*
  For a.e. $x \in \Omega$,

$$\int_Y a(x,\cdot) \; \Big[\nabla p_0(x) + \nabla_y p_1(x,\cdot)\Big] \nabla_y \psi = \int_Y \theta_0(x,\cdot) \; b(x,\cdot) \frac{\partial \psi}{\partial y_1}, \tag{1.10}$$

for every $\psi \in H^1_\sharp(Y)$.

Proof. Using the test function

$$\phi(x) + \varepsilon \; \phi_1(x) \psi\left(\frac{x}{\varepsilon}\right)$$

with $\phi \in V_0$, $\phi_1 \in \mathcal{D}(\Omega)$ and $\psi \in H^1_\sharp(Y)$ in problem $(\mathcal{P}^\varepsilon_\theta)$, one has:

$$\begin{aligned}\int_\Omega \; a\left(x, \frac{x}{\varepsilon}\right) \nabla p_\varepsilon(x) \Big[\nabla\phi(x) + \phi_1(x) \nabla_y \psi\left(\frac{x}{\varepsilon}\right) + \varepsilon\psi\left(\frac{x}{\varepsilon}\right) \nabla_x \phi_1(x)\Big] \; dx \\ = \int_\Omega \theta_\varepsilon(x) b\left(x, \frac{x}{\varepsilon}\right) \left[\frac{\partial \phi}{\partial x_1}(x) + \phi_1(x) \frac{\partial \psi}{\partial y_1}\left(\frac{x}{\varepsilon}\right) + \varepsilon\psi\left(\frac{x}{\varepsilon}\right) \frac{\partial \phi_1}{\partial x_1}(x)\right] \; dx.\end{aligned}$$

Passing to the limit ($\varepsilon \to 0$) gives us the macroscopic equation (with $\phi_1 \equiv 0$) and the microscopic equation (with $\phi \equiv 0$), using density results. □

Let us define the local problems, respectively denoted $(\mathcal{M}^\star_i)$, $(\mathcal{N}^\star_i)$ and $(\mathcal{N}^0_i)$:

Find $W^\star_i$, $\chi^\star_i$, χ^0_i $(i = 1, 2)$ in $L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$, such that, for a.e. $x \in \Omega$:

$$\int_Y \; a(x,\cdot) \; \nabla_y W^\star_i(x,\cdot) \; \nabla_y \psi \;\; = \;\; \int_Y \; a(x,\cdot) \; \frac{\partial \psi}{\partial y_i}, \tag{1.11}$$

$$\int_Y a(x,\cdot)\ \nabla_y \chi_i^\star(x,\cdot)\ \nabla_y \psi = \int_Y b(x,\cdot)\ \frac{\partial \psi}{\partial y_i}, \tag{1.12}$$

$$\int_Y a(x,\cdot)\ \nabla_y \chi_i^0(x,\cdot)\ \nabla_y \psi = \int_Y \theta_0(x,\cdot)\ b(x,\cdot)\ \frac{\partial \psi}{\partial y_i}, \tag{1.13}$$

for all $\psi \in H^1_\sharp(Y)$. We immediatly have:

Lemma 1.15. *Problem $(\mathcal{M}_i^\star)$ (resp. $(\mathcal{N}_i^\star)$,$(\mathcal{N}_i^0)$) admits a unique solution $W_i^\star$ (resp. $\chi_i^\star$, χ_i^0) in $L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$.*

Theorem 1.16. *The homogenized problem can be written*

$$(\mathcal{P}_\theta^\star)\begin{cases} \textit{Find } (p_0,\Theta_1,\Theta_2) \in V_a \times L^\infty(\Omega) \times L^\infty(\Omega) \textit{ such that} \\ \displaystyle\int_\Omega \mathcal{A}\cdot\nabla p_0\ \nabla\phi = \int_\Omega \underline{b}^0\ \nabla\phi, \quad \forall\ \phi \in V_0, \\ p_0 \geq 0 \quad \textit{and} \quad p_0\ (1-\Theta_i) = 0,\ (i=1,2) \quad \textit{a.e.}, \end{cases}$$

with $\mathcal{A} = \begin{pmatrix} a_{11}^\star & a_{12}^\star \\ a_{21}^\star & a_{22}^\star \end{pmatrix}$, $\underline{b}^0 = \begin{pmatrix} \Theta_1 b_1^\star \\ \Theta_2 b_2^\star \end{pmatrix}$ *and* $\widetilde{f}(x) = \displaystyle\int_Y f(x,y)\ dy$, *being the homogenized coefficients defined as*

$$a_{ij}^\star = \widetilde{a}\ \delta_{ij} - \widetilde{\left[a\frac{\partial W_j^\star}{\partial y_i}\right]}, \quad b_i^\star = \widetilde{b} - \widetilde{\left[a\frac{\partial \chi_i^\star}{\partial y_i}\right]}.$$

Moreover, the homogenized problem admits at least a solution.

Proof. From Lemma 1.15, one has:

$$p_1(x,y) = -W^\star(x,y)\cdot\nabla p_0(x) + \chi_1^0(x,y), \quad \text{in } L^2(\Omega; H^1_\sharp(Y)/\mathbb{R}) \tag{1.14}$$

with $W^\star = \begin{pmatrix} W_1^\star \\ W_2^\star \end{pmatrix}$. Let us notice that $\chi_1^0(x,y)$ depends on $\theta_0(x,y)$ which is unknown. Using Equation (1.14) in the macroscopic equation gives:

$$\begin{aligned} \int_\Omega \left[\widetilde{a}\ I - \widetilde{a\nabla W^\star}\right]\cdot\nabla p_0\ \nabla\phi &= \int_\Omega \left[\ \widetilde{(\theta_0 b)} - \widetilde{\left(a\frac{\partial \chi_1^0}{\partial y_1}\right)}\ \right]\frac{\partial\phi}{\partial x_1} \\ &+ \int_\Omega \left[\ -\widetilde{\left(a\frac{\partial \chi_1^0}{\partial y_2}\right)}\ \right]\frac{\partial\phi}{\partial x_2}, \end{aligned} \tag{1.15}$$

for every $\phi \in V_0$. Introducing the notations $b_i^0 = \widetilde{(\theta_0 b)} - \widetilde{\left(a\dfrac{\partial \chi_1^0}{\partial y_i}\right)}$ $(i=1,2)$, one gets

$$\int_\Omega \mathcal{A}\cdot\nabla p_0\ \nabla\phi = \int_\Omega \underline{b}^0\nabla\phi, \quad \forall\ \phi \in V_0,$$

with

$$\mathcal{A} = \widetilde{a}\ I - \widetilde{a\nabla W^\star} \qquad \underline{b}^0 = \begin{pmatrix} b_1^0 \\ b_2^0 \end{pmatrix}.$$

Introducing the ratios $\Theta_i = b_i^0/b_i^\star$ in the vector $\underline{b}^0$ concludes the proof. □

Remark 1.17. *The homogenized lubrication problem can be considered as a generalized Reynolds-type problem with two saturation functions Θ_i ($i = 1, 2$). Let us notice that if there is no cavitation phenomena (i.e. $p_0 > 0$) then $\Theta_i = 1$: thus, we get the classical homogenized Reynolds equation (without cavitation) (see [BC89]). But several aspects remain hard to describe:*

(a) The homogenized problem leads us to consider two different saturation functions, since an extra term has to be added (in the x_2 direction of the flow) when comparing the homogenized problem to the initial problem.

(b) Another point is that the property $0 \leq \Theta_i \leq 1$ is missing, i.e. we cannot guarantee that homogenized cavitation parameters are smaller than 1 in cavitation areas !

(c) We are not able to prove any uniqueness result, for the homogenized problem, using the methods described in Section 1.1.

(d) Algorithms are known to solve the smooth problem (see for instance the papers by Alt [Alt81], Bayada, Chambat and Vázquez [BCV98], Marini and Pietra [MP86]). But how to solve the homogenized problem numerically ? How to treat the two different saturation functions?

Thus these four difficulties have to be underlined in the most general case and, in the following subsections, we show how it is possible to solve them, fully or at least partially. Additional assumptions have to be made in order to get an homogenized problem with a structure which is similar to the initial one. This will be the subject of the following subsection. Before starting this study, let us conclude this subsection with the following theorem:

Theorem 1.18. *The homogenized problem ($\mathcal{P}_\theta^\star$) admits a solution (p_0, Θ, Θ) satisfying the property $0 \leq \Theta \leq 1$ a.e.*

Proof. The result is obtained in three steps: first, we consider the penalized rough problem ($\mathcal{P}_\eta^\varepsilon$); then, we apply the homogenization process to the penalized problem (i.e. $\varepsilon \to 0$); finally, we pass to the limit on the penalization parameter (i.e. $\eta \to 0$).

■ *1st step* - Let us consider the rough penalized problem:

$$(\mathcal{P}_\eta^\varepsilon)\begin{cases} \textit{Find } p_\varepsilon^\eta \in V_a \textit{ such that:} \\ \displaystyle\int_\Omega a_\varepsilon \ \nabla p_\varepsilon \ \nabla\phi = \int_\Omega H_\eta(p_\varepsilon^\eta) \ b_\varepsilon \ \frac{\partial \phi}{\partial x_1}, \quad \forall \ \phi \in V_0. \\ p_\varepsilon \geq 0, \quad a.e. \end{cases}$$

■ *2nd step* - Similarly to the exact rough problem, we get a priori estimates on the pressure, i.e. $\|p_\varepsilon^\eta\|_{H^1(\Omega)} \leq C_3$ where C_3 only depends on Ω. From the previous estimate, we deduce that there exists $p_0^\eta \in V_a$ ($p_0^\eta \geq 0$ a.e.) such that, up to a subsequence, p_ε^η weakly converges to p_0^η in $H^1(\Omega)$. Moreover, p_ε^η two-scale converges to p_0^η and there exists $p_1^\eta \in L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$ and a subsequence ε' still denoted ε such that $\nabla p_\varepsilon^\eta$ two-scale converges to $\nabla p_0^\eta + \nabla_y p_1^\eta$. Then, with the two-scale homogenization technique, we get the following macro/microscopic decomposition:

• Macroscopic equation:

$$\int_\Omega \Big(\int_Y a\Big[\nabla p_0^\eta + \nabla_y p_1^\eta\Big]\Big) \ \nabla \phi = \int_\Omega \Big(\int_Y H_\eta(p_0^\eta) \ b\Big) \frac{\partial \phi}{\partial x_1}, \tag{1.16}$$

for every ϕ in V_0.

• Microscopic equation:

For a.e. $x \in \Omega$,

$$\int_Y a(x,\cdot) \ \Big[\nabla p_0^\eta(x) + \nabla_y p_1^\eta(x,\cdot)\Big]\nabla_y \psi = \int_Y H_\eta(p_0^\eta(x)) \ b(x,\cdot) \ \frac{\partial \psi}{\partial y_1}, \tag{1.17}$$

for every $\psi \in H^1_\sharp(Y)$.

Then introducing the local problems defined in Equations (1.11) and (1.12), we get:

$$p_1^\eta(x,y) = -W^\star(x,y)\cdot \nabla p_0^\eta(x) + H_\eta(p_0^\eta(x)) \ \chi_1^\star(x,y), \quad \text{in } L^2(\Omega; H^1_\sharp(Y)/\mathbb{R}). \tag{1.18}$$

Using Equation (1.18) in the macroscopic equation gives:

$$\begin{aligned} \int_\Omega \Big[\widetilde{a} \ I - \widetilde{a\nabla W^\star}\Big] \ \nabla p_0^\eta \ \nabla \phi \ &= \ \int_\Omega H_\eta(p_0^\eta) \ \Big[\ \ \widetilde{b} - \widetilde{\Big(a\frac{\partial \chi_1^\star}{\partial y_1}\Big)} \ \ \Big] \ \frac{\partial \phi}{\partial x_1} \\ &+ \ \int_\Omega H_\eta(p_0^\eta) \ \Big[\ \ \ \ -\widetilde{\Big(a\frac{\partial \chi_1^\star}{\partial y_2}\Big)} \ \ \Big] \ \frac{\partial \phi}{\partial x_2}, \end{aligned} \tag{1.19}$$

for every $\phi \in V_0$. Then, using the definitions of $b_i^\star$ ($i = 1,2$) (see Theorem 1.16) and introducing vector $\underline{b}^\star$ whose i-th component is $b_i^\star$, the homogenized penalized problem can be written as

$$(\mathcal{P}_\eta^\star)\begin{cases} \textit{Find } p_0^\eta \in V_a \textit{ such that} \\ \displaystyle\int_\Omega \mathcal{A} \cdot \nabla p_0^\eta \, \nabla \phi = \int_\Omega H_\eta(p_0^\eta) \, \underline{b}^\star \, \nabla \phi, \quad \forall \, \phi \in V_0, \\ p_0^\eta \geq 0, \quad a.e. \end{cases}$$

■ *3rd step* - As $\mathcal{A}$ is a coercive matrix (see [BLP78]), we establish a priori estimates on p_0^η, in the $H^1(\Omega)$ norm, which do not depend on η, so that there exists $p_0 \in V_a$, ($p_0 \geq 0$ a.e.) and $\Theta \in L^\infty(\Omega)$ such that

$$\begin{array}{rcl} p_0^\eta & \rightharpoonup & p_0, \quad \text{in } H^1(\Omega), \\ H_\eta(p_0^\eta) & \rightharpoonup & \Theta, \quad \text{in } L^\infty(\Omega) \text{ weak-}\star. \end{array}$$

Passing to the limit ($\eta \to 0$) in problem ($\mathcal{P}_\eta^\star$) concludes the proof, since the properties $0 \leq \Theta \leq 1$ and $p_0 \, (1 - \Theta) = 0$ a.e. are classically obtained as in Section 1.1. □

Remark 1.19. *Let us recall that we are not able to prove a uniqueness result on the general problem. But we can wonder whether it is possible to obtain a uniqueness result among the class of solutions* $(p_0, \Theta_1, \Theta_2)$ *satisfying* $\Theta_1 = \Theta_2 = \Theta$ *with* $0 \leq \Theta \leq 1$ *(and, of course,* $p_0 \geq 0$*,* $p_0 \, (1 - \Theta) = 0$*). In fact, it is not possible to get such a result using the method described in Section 1.1, because it is not well-suited to a flow whose component in the* x_2 *direction is different from 0.*

Remark 1.20. *Theorem 1.18 guarantees that we are able to build an homogenized problem with isotropic saturation from the penalized problem, although it is not the case when directly studying the homogenization of the exact problem (in the most general case):*

- *the penalized problem allows us to build a solution in pressure/saturation* (p_0, Θ, Θ) *where the saturation* Θ *satisfies* $0 \leq \Theta \leq 1$ *(and, also,* $p_0 \geq 0$ *and* $p_0 \, (1 - \Theta) = 0$*);*
- *by contrast, the exact problem with the homogenization process builds a solution in pressure / double-saturation* $(p_0, \Theta_1, \Theta_2)$ *for which we are not able to conclude that* $0 \leq \Theta_i \leq 1$ *(although the following properties hold:* $p_0 \geq 0$ *and* $p_0 \, (1 - \Theta_i) = 0$*,* $(i = 1, 2)$*).*

At that point, it is important to know whether $\theta_0(x, y)$ *depends on* y *or not: that* θ_0 *does not depend on the* y *variable would mean that the homogenized exact problem and the homogenized penalized problem (after passing to the limit on* η*) are identical, i.e. saturation phenomena would be isotropic. More precisely, in the exact homogenized problem, such an assumption leads us to* $\Theta_1 = \Theta_2 = \theta_0$ *(see Equations (1.13) and (1.19)),* $0 \leq \Theta_i = \theta_0 \leq 1$ *(see Propositions 1.12 and 1.13). But, in fact, numerical tests evidence that such an assumption is not valid in general, as it will be pointed out in the next section.*

Remark 1.21. *It is now possible to find, numerically, a solution of problem* $(\mathcal{P}_\theta^\star)$*, by focusing on solutions* (p_0, Θ, Θ) *satisfying* $0 \leq \Theta \leq 1$ *(with* $p_0 \geq 0$ *and* $p_0\ (1-\Theta) = 0$*), and using algorithms that have been previously mentioned. In that prospect, it allows us to eliminate another difficulty that has been underlined in Remark 1.17. But, since we do not have any uniqueness result, we cannot guarantee that each solution* (p, Θ_1, Θ_2) *satisfies* $\Theta_1 = \Theta_2$ *and we are not able to build numerically solutions with two different saturation functions. We can neither illustrate numerically anisotropic effects on the saturation, nor prove that all the solutions have the form* (p_0, Θ, Θ)*.*

1.2.4 Some particular cases

Longitudinal and transverse roughness

Our interest in studying the behaviour of the solution when considering transverse or longitudinal roughness is highly motivated by the mechanical applications. From a mathematical point of view, we may even consider a product of transverse and longitudinal roughness i.e. we should consider, in this subsection, the following assumption:

Assumption 4.

(i) $a(x,y) = a_1(x,y_1)\ a_2(x,y_2)$,

(ii) $\exists\ m_{a,i},\ M_{a,i},\quad 0 < m_{a,i} \leq a_i \leq M_{a,i}$, $(i = 1, 2)$,

(iii) $b(x,y) = b_1(x,y_1)\ b_2(x,y_2)$,

(iv) $\exists\ m_{b,i},\ M_{b,i},\quad 0 < m_{b,i} \leq b_i \leq M_{b,i}$, $(i = 1, 2)$.

It is clear that the earlier assumption is just a separation of the microscale variables, which allows us to take into account either transverse or longitudinal roughness effects, but also particular full two dimensional roughness effects. For a dimensionless journal bearing, we may consider gaps with roughness patterns described on Fig.1.2–1.5, corresponding to a smooth gap $1 + \rho\cos(x_1)$, $x \in]0, 2\pi[\times]0, 1[$.

Lemma 1.22. *Under Assumption 4, it follows that:*

$$\mathcal{A} = \begin{pmatrix} \dfrac{\widetilde{a_2}}{\widetilde{a_1^{-1}}} & 0 \\ 0 & \dfrac{\widetilde{a_1}}{\widetilde{a_2^{-1}}} \end{pmatrix}.$$

Proof.

■ *Diagonal terms of the matrix.* For this, let us recall the variational formulation (see

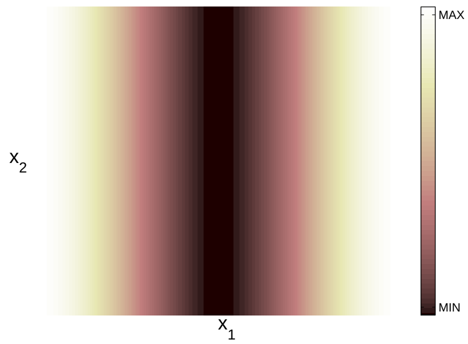


Figure 1.2. Normalized gap (no roughness patterns)

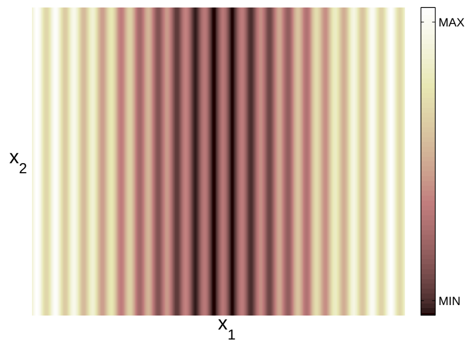


Figure 1.3. Normalized gap with transverse roughness patterns

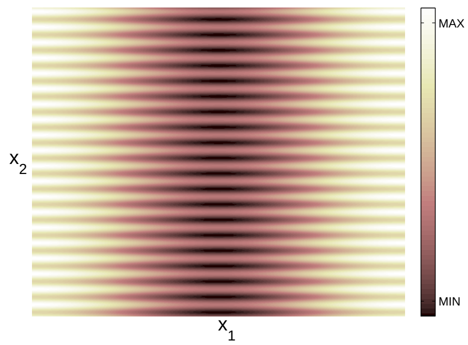


Figure 1.4. Normalized gap with longitudinal roughness patterns

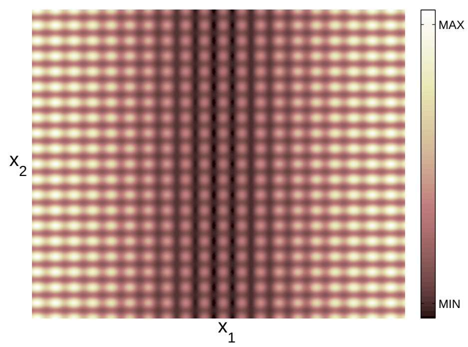


Figure 1.5. Normalized gap with two dimensional roughness patterns

Equation (1.11)) of problem $(\mathcal{M}_i^\star)$ $(i = 1, 2)$:

$$\int_Y a\ \nabla_y W_i^\star\ \nabla_y \psi = \int_Y a\ \frac{\partial \psi}{\partial y_i}, \quad \forall \psi \in H^1_\sharp(Y).$$

Let $j \in \{1, 2\}$, with $j \neq i$. Denoting $\Big[f\Big]_{Y_j}$ the averaging process of a function f on the y_j variable and using a test function only depending on y_i, one has:

$$\int_{Y_i} \Big[a\ \frac{\partial W_i^\star}{\partial y_i}\Big]_{Y_j} \frac{d\psi}{dy_i} = \int_{Y_i} \Big[a\Big]_{Y_j} \frac{d\psi}{dy_i}, \quad \forall \psi \in H^1_\sharp(Y_i)$$

Then, one has, for a.e. $x \in \Omega$, that:

$$\Big[a\ \frac{\partial W_i^\star}{\partial y_i}\Big]_{Y_j} = \Big[a\Big]_{Y_j} + C_{ii}(x). \tag{1.20}$$

Using Assumption 4 and dividing Equation (1.20) by a_i, we have:

$$\Big[a_j\ \frac{\partial W_i^\star}{\partial y_i}\Big]_{Y_j} = \Big[a_j\Big]_{Y_j} + \frac{C_{ii}}{a_i}.$$

Now, averaging on the y_j variable and using the Y periodicity of $W_i^\star$ give us

$$0 = \widetilde{a_j} + C_{ii}\ \widetilde{a_i^{-1}},$$

so that $C_{ii}(x) = -\dfrac{\widetilde{a_j}}{\widetilde{a_i^{-1}}}$. Moreover, using the definition of $\mathcal{A}_{ii}$ (see Theorem 1.16) and Equation (1.20), one has $C_{ii}(x) = -\mathcal{A}_{ii}(x)$, so that

$$\mathcal{A}_{ii}(x) = \frac{\widetilde{a_j}}{\widetilde{a_i^{-1}}}(x), \quad i = 1, 2.$$

■ *Non-diagonal terms of the matrix.* For this, let $i, j \in \{1, 2\}$, $j \neq i$. Recalling the variational formulation of problem $(\mathcal{M}_i^\star)$ $(i = 1, 2)$ and using a test function only depending on y_j, one has:

$$\int_{Y_j} \Big[a\ \frac{\partial W_i^\star}{\partial y_j}\Big]_{Y_i} \frac{d\psi}{dy_j} = 0, \quad \forall \psi \in H^1_\sharp(Y_j).$$

Then, for a.e. $x \in \Omega$, we have:

$$\Big[a\ \frac{\partial W_i^\star}{\partial y_j}\Big]_{Y_i} = C_{ij}(x). \tag{1.21}$$

Using Assumption 4, dividing by a_j, averaging on the y_i variable and since $W_i^\star$ is Y periodic, we get that $C_{ij}(x) = 0$ (for $i \neq j$). Moreover, using the definition of $\mathcal{A}_{ij}$

(see Theorem 1.16) and Equation (1.21), one has $C_{ij}(x) = -\mathcal{A}_{ij}(x)$ so that $\mathcal{A}_{ij}(x) = 0$ $(i \neq j)$. □

Lemma 1.23. *Under Assumption 4, we deduce that:*

$$\underline{b}^0 = \begin{pmatrix} \Theta_1 b_1^\star \\ 0 \end{pmatrix}, \tag{1.22}$$

where the following relationships hold:

$$0 \leq \Theta_1 \leq 1 \quad \textit{and} \quad p_0\ (1 - \Theta_1) = 0 \quad \textit{a.e.}$$

Moreover, the homogenized coefficient $b_1^\star$ satisfies:

$$b_1^\star(x) = \Big[\frac{1}{\widetilde{a_1^{-1}}}\ \widetilde{\Big(\frac{b}{a_1}\Big)}\Big](x). \tag{1.23}$$

Proof. The first part of the proof lies in the determination of vector $\underline{b}_0$. In the second part, we calculate the homogenized coefficient $b_1^\star$.

■ *1st part - Computation of the components of vector $\underline{b}_0$:*

► Let us study the first term of vector $\underline{b}^0$. Thus, denoting $w_1 = \chi_1^\star - \chi_1^0$ and combining problems $(\mathcal{N}_1^\star)$ and $(\mathcal{N}_1^0)$, one gets, for a.e. $x \in \Omega$, that:

$$\int_Y a\ \nabla_y w_1\ \nabla_y \psi = \int_Y b\ (1 - \theta_0)\ \frac{\partial \psi}{\partial y_1}, \quad \forall \psi \in H^1_\sharp(Y).$$

Now, using a test function only depending on y_1, one has:

$$\int_{Y_1} \Big[a\ \frac{\partial w_1}{\partial y_1}\Big]_{Y_2} \frac{d\psi}{dy_1} = \int_{Y_1} \Big[b\ (1 - \theta_0)\Big]_{Y_2} \frac{d\psi}{dy_1}, \quad \forall \psi \in H^1_\sharp(Y_1).$$

Then, for a.e. $x \in \Omega$, we get:

$$\Big[a\ \frac{\partial w_1}{\partial y_1}\Big]_{Y_2} = \Big[b\ (1 - \theta_0)\Big]_{Y_2} + C(x), \tag{1.24}$$

where $C(x)$ is an additive constant only depending on x. Next, using Assumption 4, dividing by a_1, averaging on the y_2 variable and using the Y periodicity of w_1, we deduce the following equality

$$\left[\widetilde{(1-\theta_0)\frac{b}{a_1}}\right] + C(x)\ \widetilde{a_1^{-1}} = 0.$$

Now, from Proposition 1.12 and Assumption 4, it is easy to get $C(x) \leq 0$. Then, averaging Equation (1.24) on the y_1 variable, we obtain that

$$\widetilde{(\theta_0 b)} - \widetilde{\left(a\frac{\partial \chi_1^0}{\partial y_1}\right)} \leq \widetilde{(b)} - \widetilde{\left(a\frac{\partial \chi_1^\star}{\partial y_1}\right)}, \qquad \text{i.e.} \qquad b_1^0 \leq b_1^\star.$$

Next, applying the earlier method to the variational formulation of problem $(\mathcal{N}_1^0)$, it is easy to conclude $0 \leq b_1^0$ $(i = 1, 2)$.

▶ Let us now study the second term of vector $\underline{b}^0$. Applying the same mehod (as earlier) to the variational formulation of problem $(\mathcal{N}_1^0)$, one has:

$$\int_{Y_2} \left[a\frac{\partial \chi_1^0}{\partial y_2}\right]_{Y_1} \frac{d\psi}{dy_2} = 0, \quad \forall \psi \in H_\sharp^1(Y_2).$$

Then, one gets:

$$\left[a\frac{\partial \chi_1^0}{\partial y_2}\right]_{Y_1} = 0, \quad \text{in } H_\sharp^1(Y_2)/\mathbb{R}.$$

From the previous equality, one obtains:

$$\left[a\frac{\partial \chi_1^0}{\partial y_2}\right]_{Y_1} = C(x), \tag{1.25}$$

for a.e. $x \in \Omega$, where $C(x)$ is an additive constant only depending on x. Next, using Assumption 4, dividing by a_2, averaging on the y_2 variable and using the Y periodicity of χ_1^0, we get that $C(x) = 0$. So, from Equation (1.25), we deduce:

$$-\widetilde{\left(a\frac{\partial \chi_1^0}{\partial y_2}\right)} = 0, \qquad \text{i.e.} \qquad b_2^0 = 0.$$

With the earlier method applied to the variational formulation of problem $(\mathcal{N}_1^\star)$, it is easy to conclude that $b_2^\star = 0$.

Now, since we have proved that $0 \leq b_1^0 \leq b_1^\star$ and $0 = b_2^0 = b_2^\star$, using the definitions of Θ_i $(i = 1, 2)$, it is easy to conclude that Equation (1.22) and property $0 \leq \Theta_1 \leq 1$ a.e. hold. Moreover, property $p_0\ (1 - \Theta_1) = 0$ a.e. is obtained from Proposition 1.13 and the definition of Θ_1. Thus, it remains to calculate the homogenized coefficient $b_1^\star$.

■ *2nd part - Computation of $b_1^\star$:*

First, considering problem $(\mathcal{N}_1^\star)$, one gets:

$$\int_Y a\ \nabla_y \chi_1^\star\ \nabla_y \psi = \int_Y b\ \frac{\partial \psi}{\partial y_1}, \quad \forall \psi \in H^1_\sharp(Y),$$

for a.e. $x \in \Omega$. Next, using a test function only depending on y_1, one has:

$$\int_{Y_1} \Big[a \frac{\partial \chi_1^\star}{\partial y_1} \Big]_{Y_2} \frac{d\psi}{dy_1} = \int_{Y_1} \Big[b \Big]_{Y_2} \frac{d\psi}{dy_1}, \quad \forall \psi \in H^1_\sharp(Y_1).$$

Then,

$$\Big[a \frac{\partial \chi_1^\star}{\partial y_1} \Big]_{Y_2} = \Big[b \Big]_{Y_2} + C_1^\star(x), \tag{1.26}$$

for a.e. $x \in \Omega$, where $C_1^\star(x)$ is an additive constant only depending on x. Using Assumption 4, dividing by a_1, averaging on the y_2 variable and using the Y periodicity of $\chi_1^\star$, leads to the following equality:

$$\widetilde{\Big[\frac{b}{a_1} \Big]} + C_1^\star(x)\ \widetilde{a_1} = 0. \tag{1.27}$$

Next, from the definition of $b_1^\star$ (see Theorem 1.16) and Equation (1.26), we deduce that $C_1^\star(x) = -b_1^\star(x)$ so that, from Equation (1.27), we conclude the proof. □

Lemma 1.24. *Under Assumption 4, it follows that*

$$\Theta_1(x) = \left[\frac{1}{\widetilde{\Big(\frac{b}{a_1} \Big)}} \ \widetilde{\Big(\frac{\theta_0 b}{a_1} \Big)} \right](x). \tag{1.28}$$

Proof. Notice that b_1^0 can be calculated by using the same method which allowed us to obtain $b_1^\star$ in the proof of Lemma 1.23, just replacing problem $(\mathcal{N}_1^\star)$, by problem $(\mathcal{N}_1^0)$. Then, we have

$$b_1^0(x) = \frac{1}{\widetilde{a_1^{-1}}} \widetilde{\Big[\frac{\theta_0 b}{a_1} \Big]}(x). \tag{1.29}$$

Definition of Θ_1 (see Theorem 1.16), Equations (1.23) and (1.29) conclude the proof. □

To summarize the previous results, we establish the following homogenized problem:

Theorem 1.25. *Under Assumption 4, the homogenized problem is:*

$$(\mathcal{P}_\theta^\star) \begin{cases} \textit{Find } (p_0,\ \Theta) \in V_a \times L^\infty(\Omega) \textit{ such that:} \\ \displaystyle\int_\Omega \mathcal{A} \cdot \nabla p_0\ \nabla \phi = \int_\Omega \Theta\ b_1^\star\ \frac{\partial \phi}{\partial x_1}, \quad \forall\ \phi \in V_0 \\ p_0 \geq 0, \quad p_0\ (1 - \Theta) = 0, \quad 0 \leq \Theta \leq 1, \quad a.e. \end{cases}$$

with the following homogenized coefficients:

$$\mathcal{A} = \begin{pmatrix} \dfrac{\widetilde{a_2}}{\widetilde{a_1^{-1}}} & 0 \\ 0 & \dfrac{\widetilde{a_1}}{\widetilde{a_2^{-1}}} \end{pmatrix}, \quad b_1^\star(x) = \Big[\frac{1}{\widetilde{a_1^{-1}}}.\widetilde{\Big(\frac{b}{a_1}\Big)}\Big](x)$$

Moreover $(\mathcal{P}_\theta^\star)$ *admits at least* (p_0, Θ) *as a solution, where*

$$\Theta(x) = \Big[\frac{1}{\widetilde{\Big(\dfrac{b}{a_1}\Big)}}.\widetilde{\Big(\frac{\theta_0 b}{a_1}\Big)}\Big](x) \tag{1.30}$$

and (p_0, θ_0) *is the two-scale limit of* $(p_\varepsilon, \theta_\varepsilon)$ *(solution of problem* $(\mathcal{P}_\theta^\varepsilon)$*).*

Remark 1.26. *In the lubrication problem, Assumption 4 implies that the gap between the two sufaces is described by the function:*

$$h\left(x, \frac{x}{\varepsilon}\right) = h_1\left(x, \frac{x_1}{\varepsilon}\right)\ h_2\left(x, \frac{x_2}{\varepsilon}\right)$$

In this case, the homogenized coefficients are the following ones:

$$\mathcal{A} = \begin{pmatrix} \dfrac{\widetilde{h_2^3}}{\widetilde{h_1^{-3}}} & 0 \\ 0 & \dfrac{\widetilde{h_1^3}}{\widetilde{h_2^{-3}}} \end{pmatrix}, \qquad b_1^\star(x) = \Big[\frac{\widetilde{h_1^{-2}}}{\widetilde{h_1^{-3}}}\ \widetilde{h_2}\Big](x)$$

and we get the precise link between the microscopic cavitation and the macroscopic cavitation, i.e.

$$\Theta(x) = \Big[\frac{1}{\widetilde{h_2}\widetilde{h_1^{-2}}}.\widetilde{\Big(\frac{\theta_0 h_2}{h_1^2}\Big)}\Big](x) \tag{1.31}$$

Theorem 1.27.
(i) Under Assumption 4, problem $(\mathcal{P}_\theta^\star)$ *admits at least a solution* (p_0, Θ)*. Moreover, the pressure* p_0 *is unique, and if there exists a set of positive measure where* $p_0(x_1, x_2) > 0$*, for any* $x_2 > 0$*, then the saturation* Θ *is unique.*
(ii) If $b^\star$ *can be written under the form* $b^\star(x_1, x_2) = b_1^\star(x_1) b_2^\star(x_2)$*, problem* $(\mathcal{P}_\theta^\star)$ *admits a unique solution.*

Proof. For (i), existence of a solution is stated in Theorem 1.25, by means of construction via the two-scale convergence techniques. Uniqueness of the pressure and, under the additional assumption, of the saturation is obtained as in Theorem 1.6. For (ii), the result is obtained as in Corollary 1.7. □

Remark 1.28. *A primal "naive" attempt leading to the homogenized problem would be to determine an equation satisfied by the weak limits of $(p_\varepsilon, \theta_\varepsilon)$, namely $(p_0, \widetilde{\theta}_0)$. Interestingly, the weak limit of the pressure does appear in the homogenized problem, but the macroscopic homogenized saturation Θ is a modified average of θ_0, weighted by the roughness effects through the influence of functions h_i.*

It is interesting to notice that Assumption 4 allows us to solve the four difficulties that we could not overcome in the most general case (see Remark 1.17). In particular, there is one single saturation function; the homogenized problem can be numerically solved using algorithms applied to the smooth problem; and it is easy, under additional realistic assumptions, to obtain a uniqueness result on both pressure and saturation. Moreover, Assumption 4 includes some important particular cases in terms of mechanical applications: transverse and longitudinal roughness. The results are easily deduced from Theorem 1.27 and given, in the next results, for a strong formulation.

Corollary 1.29. *If h does not depend on y_2 (transverse roughness), then the homogenized problem can be written as:*

$$\begin{cases} \dfrac{\partial}{\partial x_1}\Big[\ \dfrac{1}{\widetilde{h^{-3}}}\ \dfrac{\partial p_0}{\partial x_1}\ \Big] + \dfrac{\partial}{\partial x_2}\Big[\ \widetilde{h^3}\ \dfrac{\partial p_0}{\partial x_2}\ \Big] = \dfrac{\partial}{\partial x_1}\Big[\ \Theta\ \dfrac{\widetilde{h^{-2}}}{\widetilde{h^{-3}}}\ \Big], & x \in \Omega, \\ p_0(x) \geq 0, \quad p_0(x)\ (1-\Theta(x)) = 0, \quad 0 \leq \Theta(x) \leq 1, & x \in \Omega, \end{cases}$$

with the following boundary conditions:

$$\begin{array}{ll} p_0 = 0 \ \text{ on } \Gamma_0 \text{ and } p_0 = p_a \text{ on } \Gamma_a & \text{(Dirichlet conditions)} \\ \Theta\ \dfrac{\widetilde{h^{-2}}}{\widetilde{h^{-3}}} - \dfrac{1}{\widetilde{h^{-3}}}\ \dfrac{\partial p_0}{\partial x_1} \text{ and } p_0 \text{ are } 2\pi x_1 \text{ periodic} & \text{(periodic conditions)} \end{array}$$

Corollary 1.30. *If h does not depend on y_1 (longitudinal roughness), then the homogenized problem can be written as:*

$$\begin{cases} \dfrac{\partial}{\partial x_1}\Big[\ \widetilde{h^3}\ \dfrac{\partial p_0}{\partial x_1}\ \Big] + \dfrac{\partial}{\partial x_2}\Big[\ \dfrac{1}{\widetilde{h^{-3}}}\ \dfrac{\partial p_0}{\partial x_2}\ \Big] = \dfrac{\partial}{\partial x_1}\Big[\ \Theta\ \widetilde{h}\ \Big], & x \in \Omega, \\ p_0(x) \geq 0, \quad p_0(x)\ (1-\Theta(x)) = 0, \quad 0 \leq \Theta(x) \leq 1, & x \in \Omega, \end{cases}$$

with the following boundary conditions:

$$\begin{array}{ll} p_0 = 0 \ \text{ on } \Gamma_0 \text{ and } p_0 = p_a \text{ on } \Gamma_a & \text{(Dirichlet conditions)} \\ \Theta\ \widetilde{h} - \widetilde{h^3}\ \dfrac{\partial p_0}{\partial x_1} \text{ and } p_0 \text{ are } 2\pi x_1 \text{ periodic} & \text{(periodic conditions)} \end{array}$$

Under Assumption 4, the homogenized problem is similar to the ε dependent one, since there is one single saturation function. This assumption, imposing a particular form of the roughness, seems to be strong but it allows us to take into account some two dimensional roughness effects. Moreover, it is somewhat surprising to see that passing from the classical homogenized equation (without cavitation) (see [BC89]) to the one obtained in our paper (including cavitation) only needs to introduce a saturation in the right-hand side; in other words, comparing the homogenized Reynolds equations - with or without cavitation -, the homogenized coefficients are not modified, although the Elrod-Adams model introduces a strong nonlinearity through the saturation function and its properties.

In the next subsubsection, we deal with oblique roughness. Obviously, this case does not fall into Assumption 4 which enables us to completely overcome the mentioned difficulties stated in the general case. However, it seems that a change of variables could allow us to recover a structure in which Assumption 4 is satisfied. We will see that it is not really the case and that the change of variables will introduce additional terms which are not fully controlled by the homogenization process; nevertheless, it allows us to define, in a rigorous way, two homogenized saturation functions, thus describing anisotropic phenomena on the cavitation. This structure can be considered as an intermediary one between the general case and the microvariables separation case.

Oblique roughness

Let us consider the mapping $\mathcal{F}_\gamma$ defined as:

$$\begin{array}{cccc} \mathcal{F}_\gamma & : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & x & \longrightarrow & X = \mathcal{F}_\gamma(x) \end{array}, \text{ with } \left\{ \begin{array}{ll} X_1^\gamma(x) = & \cos\gamma\ x_1 + \sin\gamma\ x_2 \\ X_2^\gamma(x) = & -\sin\gamma\ x_1 + \cos\gamma\ x_2 \end{array} \right.$$

We suppose that the effective gap can be described as follows:

Assumption 5. *For a given angle γ, let be h_ε a function such that*

$$\forall x \in \Omega, \quad h_\varepsilon(x) = h_1\left(x, \frac{X_1^\gamma(x)}{\varepsilon}\right)\ h_2\left(x, \frac{X_2^\gamma(x)}{\varepsilon}\right),$$

with $0 < m_i^0 \leq h_i \leq M_i^0$ *a.e.* $(i = 1, 2)$.

Obviously, functions satisfying Assumption 5 (see for instance FIG.1.6) do not satisfy Assumption 4 (except for particular values of γ). Let us drop the overscripts γ (for the sake of simplicity). Now, we say that $x = (x_1, x_2)$ (*resp.* $X = (X_1, X_2)$) denotes the original (*resp.* new) spatial coordinates. So, introducing the vector $e_{-\gamma} = (\cos\gamma, -\sin\gamma)$,

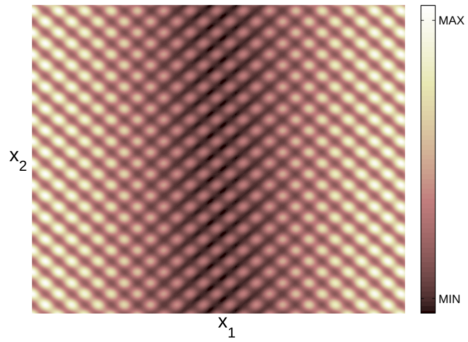


Figure 1.6. Normalized gap with oblique roughness patterns

problem $(\mathcal{P}_\theta^\varepsilon)$ can be described in the X coordinates as follows:

$$\left(\breve{\mathcal{P}_\theta}^{\varepsilon}\right)\begin{cases} \textit{Find } (\breve{p}_\varepsilon, \breve{\theta}_\varepsilon) \in \breve{V}_a \times L^\infty(\breve{\Omega}) \textit{ such that:} \\ \displaystyle\int_{\breve{\Omega}} \breve{h}_\varepsilon^3(X)\ \nabla\breve{p}_\varepsilon(X)\ \nabla\phi(X)\ dX = \int_{\breve{\Omega}} \breve{\theta}_\varepsilon(X)\ \breve{h}_\varepsilon(X)\ e_{-\gamma}\ \nabla\phi(X)\ dX, \quad \forall\ \phi \in \breve{V}_0, \\ \breve{p}_\varepsilon \geq 0, \quad \breve{p}_\varepsilon\ (1-\breve{\theta}_\varepsilon) = 0, \quad 0 \leq \breve{\theta}_\varepsilon \leq 1, \quad \textit{a.e.,} \end{cases}$$

where $\breve{f}(X) = f(x)$ and $\breve{\Omega} = \mathcal{F}_\gamma(\Omega)$, with the following functional spaces:

$$\begin{array}{rcl} \breve{V}_a & = & \Big\{\phi \in H^1(\breve{\Omega}),\ \phi_{|_{\breve{\Gamma}_l}} = \phi_{|_{\breve{\Gamma}_r}},\ \phi_{|_{\breve{\Gamma}_0}} = 0,\ \phi_{|_{\breve{\Gamma}_a}} = \breve{p}_a\Big\}, \\ \breve{V}_0 & = & \Big\{\phi \in H^1(\breve{\Omega}),\ \phi_{|_{\breve{\Gamma}_l}} = \phi_{|_{\breve{\Gamma}_r}},\ \phi_{|_{\breve{\Gamma}_0}} = 0,\ \phi_{|_{\breve{\Gamma}_a}} = 0\Big\}, \end{array}$$

where Γ_l (*resp.* Γ_r) denotes the left (*resp.* right) lateral boundary.

Remark 1.31. *In the new coordinates, one has*

$$\breve{h}_\varepsilon(X) = \breve{h}_1\Big(X, \frac{X_1}{\varepsilon}\Big)\ \breve{h}_2\Big(X, \frac{X_2}{\varepsilon}\Big).$$

From now on, we denote $\breve{a}_i(X, y_i) = \breve{h}_i^3(X, y_i)$ *and* $\breve{b}_i(X, y_i) = \breve{h}_i(X, y_i)$ *(*$i = 1, 2$*). Then* $\breve{a}(X, y) = \breve{a}_1(X, y_1)\ \breve{a}_2(X, y_2)$ *and* $\breve{b}(X, y) = \breve{b}_1(X, y_1)\ \breve{b}_2(X, y_2)$ *satisfy Assumption 4 in the* X *coordinates.*

Remark 1.32. *The formulation of the lubrication problem in the new coordinates system is equivalent to a generalized Reynolds problem as it happens with an oblique flow direction* $e_{-\gamma} = (\cos\gamma, -\sin\gamma)$*, instead of* $e = (1, 0)$ *in the classical one.*

Theorem 1.33. *We have the following convergences:*

(i) There exists $\breve{p}_0 \in H^1(\breve{\Omega})$ *such that, up to a subsequence,*

$$\breve{p}_\varepsilon \rightharpoonup \breve{p}_0, \quad \text{in } H^1(\breve{\Omega}) \qquad \text{and} \qquad \breve{p}_\varepsilon \to \breve{p}_0, \quad \text{in } L^2(\breve{\Omega}).$$

Moreover $\breve{p}_0 \in \breve{V}_a$*, and* $\breve{p}_0 \geq 0$ a.e..

(ii) $\breve{p}_\varepsilon(X)$ *two-scale converges to* $\breve{p}_0(X)$*. Moreover, there exists* $\breve{p}_1(X,y) \in L^2(\breve{\Omega}; H^1_\sharp(Y)/\mathbb{R})$ *and a subsequence* ε' *still denoted* ε *such that* $\nabla\breve{p}_\varepsilon(X)$ *two-scale converges to* $\nabla\breve{p}_0(X)+\nabla_y\breve{p}_1(X,y)$*.*

(iii) There exists $\breve{\theta}_0(X,y) \in L^2(\breve{\Omega} \times Y)$ *and a subsequence* ε" *still denoted* ε *such that* $\breve{\theta}_\varepsilon(X)$ *two-scale converges to* $\breve{\theta}_0(X,y)$*.*

Proof. The result is easily obtained after establishing a priori estimates which do not depend on ε (see Subsection 1.2.1). □

Theorem 1.34. *Under Assumption 5, one gets the following homogenized problem in the X coordinates:*

$$(\breve{\mathcal{P}}_\theta{}^\star)\begin{cases} \text{Find } (\breve{p}_0,\ \breve{\Theta}_1,\ \breve{\Theta}_2) \in \breve{V}_a \times L^\infty(\breve{\Omega}) \times L^\infty(\breve{\Omega}) \text{ such that:} \\ \displaystyle\int_{\breve{\Omega}} \breve{\mathcal{A}}(X)\cdot\nabla\breve{p}_0(X)\ \nabla\phi(X)\ dX = \int_{\breve{\Omega}} \breve{\mathcal{B}}^0(X)\cdot e_{-\gamma}\ \nabla\phi(X)\ dX, \quad \forall\ \phi \in \breve{V}_0, \\ \breve{p}_0 \geq 0, \quad \breve{p}_0\ (1-\breve{\Theta}_i) = 0, \quad 0 \leq \breve{\Theta}_i \leq 1, \quad (i=1,2) \quad a.e., \end{cases}$$

with the following expressions:

$$\breve{\mathcal{A}} = \begin{pmatrix} \dfrac{\widetilde{\breve{a}_2}}{\widetilde{\breve{a}_1^{-1}}} & 0 \\ 0 & \dfrac{\widetilde{\breve{a}_1}}{\widetilde{\breve{a}_2^{-1}}} \end{pmatrix}, \quad \breve{\mathcal{B}}^0 = \begin{pmatrix} \breve{\Theta}_1\ \breve{b}_1^\star & 0 \\ 0 & \breve{\Theta}_2\ \breve{b}_2^\star \end{pmatrix},$$

and

$$\breve{b}_i^\star(X) = \Big[\frac{1}{\widetilde{\breve{a}_i^{-1}}}\ \widetilde{\Big(\frac{\breve{b}}{\breve{a}_i}\Big)}\Big](X), \quad i=1,2.$$

Moreover problem $(\breve{\mathcal{P}}_\theta{}^\star)$ *admits* $(\breve{p}_0,\ \breve{\Theta}_1,\ \breve{\Theta}_2)$ *as a solution, where*

$$\breve{\Theta}_i(X) = \Big[\frac{1}{\widetilde{\Big(\dfrac{\breve{b}}{\breve{a}_i}\Big)}}.\widetilde{\Big(\frac{\breve{\theta}_0\breve{b}}{\breve{a}_i}\Big)}\Big](X), \quad i=1,2$$

and $(\breve{p}_0,\breve{\theta}_0)$ *is the two-scale limit of* $(\breve{p}_\varepsilon,\breve{\theta}_\varepsilon)$ *(solution of problem* $(\breve{\mathcal{P}}_\theta{}^\varepsilon)$*).*

Proof. We use the same techniques as before, the only modification comes from the presence of an additional term in the right-hand side of the equation. We briefly sketch the main steps of the complete proof:

■ *1st step: Properties of the two-scale limits* - Let $(\breve{p}_0, \breve{\theta}_0)$ be the two-scale limit of $(\breve{p}_\varepsilon, \breve{\theta}_\varepsilon)$ (see Theorem 1.33). Then one has:

(i) $\breve{p}_0\ (1-\breve{\theta}_0) = 0 \quad \text{in} \quad L^2(\breve{\Omega}\times Y)$,

(ii) $0 \leq \breve{\theta}_0 \leq 1 \quad$ a.e.

■ *2nd step: Macro/microscopic decomposition* - Using the classical techniques (previously used in Subsections 1.2.1 and 1.2.3), one gets:

(i) Macroscopic equation:

$$\int_{\breve{\Omega}} \Big(\int_Y \breve{a} \Big[\nabla \breve{p}_0 + \nabla_y \breve{p}_1\Big]\Big) \nabla\phi = \int_{\breve{\Omega}} \Big(\int_Y \breve{\theta}_0\ \breve{b}\Big)\ e_{-\gamma} \nabla \phi,$$

for every ϕ in $\breve{V}_0$.

(ii) Microscopic equation: for a.e. $X \in \breve{\Omega}$,

$$\int_Y \breve{a}(X,\cdot)\Big[\nabla \breve{p}_0(X) + \nabla_y \breve{p}_1(X,\cdot)\Big] \nabla_y \psi = \int_Y \breve{\theta}_0(X,\cdot)\ \breve{b}(X,\cdot)\ e_{-\gamma} \nabla_y \psi,$$

for every $\psi \in H^1_\sharp(Y)$.

■ *3rd step: Local problems and macroscopic equation* - The local problems $(\breve{\mathcal{M}}_i^\star)$, $(\breve{\mathcal{N}}_i^\star)$ and $(\breve{\mathcal{N}}_i^0)$ are identical to the ones defined in Subsection 1.2.3 (up to the notations adapted to the X coordinates). Then, one has:

$$\int_{\breve{\Omega}} \breve{\mathcal{A}} \cdot \nabla \breve{p}_0\ \nabla \phi = \int_{\breve{\Omega}} \breve{\mathcal{B}}^0 \cdot e_{-\gamma}\ \nabla\phi, \quad \forall\ \phi \in \breve{V}_0,$$

with the following notations:

$$\breve{\mathcal{A}} = \begin{pmatrix} \widetilde{\breve{a}} - \widetilde{\Big[\breve{a}\dfrac{\partial W_1^\star}{\partial y_1}\Big]} & -\widetilde{\Big[\breve{a}\dfrac{\partial W_2^\star}{\partial y_1}\Big]} \\ -\widetilde{\Big[\breve{a}\dfrac{\partial W_1^\star}{\partial y_2}\Big]} & \widetilde{\breve{a}} - \widetilde{\Big[\breve{a}\dfrac{\partial W_2^\star}{\partial y_2}\Big]} \end{pmatrix}, \qquad \breve{\mathcal{B}}^0 = \begin{pmatrix} \breve{\Theta}_{11}\ \breve{b}_{11}^\star & \breve{\Theta}_{12}\ \breve{b}_{12}^\star \\ & \\ \breve{\Theta}_{21}\ \breve{b}_{21}^\star & \breve{\Theta}_{22}\ \breve{b}_{22}^\star \end{pmatrix},$$

using the notations $(i, j = 1, 2)$:

$$\breve{b}^{\star}_{ij} = \widetilde{\breve{b}}\ \delta_{ij} - \widetilde{\left[\breve{a}\frac{\partial \chi^{\star}_j}{\partial y_i}\right]}, \qquad \breve{b}^{0}_{ij} = \widetilde{\left[\breve{\theta}_0 \breve{b}\right]}\ \delta_{ij} - \widetilde{\left[\breve{a}\frac{\partial \chi^{0}_j}{\partial y_i}\right]},$$

and defining the following ratios $(i, j = 1, 2)$:

$$\breve{\Theta}_{ij} = \frac{\breve{b}^0_{ij}}{\breve{b}^{\star}_{ij}},$$

where $W^{\star}_i$, $\chi^{\star}_i$ and χ^0_i are the solutions of the local problems $(\breve{\mathcal{M}}^{\star}_i)$, $(\breve{\mathcal{N}}^{\star}_i)$ and $(\breve{\mathcal{N}}^{0}_i)$ (consider the analogy with Equations (1.11)–(1.13)).

■ *4th step: Simplifications due to Assumption 5* - Assumption 4 in the X coordinates (issued from Assumption 5) allows us to use the same techniques as in the previous subsubsection to obtain the simplifications on $\breve{\mathcal{A}}$ and $\breve{\mathcal{B}}^0$. □

Remark 1.35. *The previous formulation is the weak formulation of a generalized Reynolds-type problem including cavitation. The main difference with the initial problem given in the formulation of* $(\breve{\mathcal{P}}_{\theta}^{\varepsilon})$ *lies in anistropic effects on the homogenized coefficients, which is a classical result in homogenization theory, but also on the saturation function.*

Theorem 1.36. *[Homogenized exact problem] Under Assumption 5, one gets the following homogenized problem in the x coordinate:*

$$(\mathcal{P}^{\star}_{\theta}) \left\{ \begin{array}{l} \textit{Find } (p_0,\ \Theta_1,\ \Theta_2) \in V_a \times L^{\infty}(\Omega) \times L^{\infty}(\Omega) \textit{ such that:} \\ \displaystyle\int_{\Omega} \mathcal{A}\cdot\nabla p_0\ \nabla\phi = \int_{\Omega} \underline{b}^0_1\ \frac{\partial \phi}{\partial x_1} + \int_{\Omega} \underline{b}^0_2\ \frac{\partial \phi}{\partial x_2}, \quad \forall\ \phi \in V_0, \\ p_0 \geq 0, \quad p_0\ (1-\Theta_i) = 0, \quad 0 \leq \Theta_i \leq 1, \quad (i=1,2) \quad a.e., \end{array}\right.$$

with the following expressions:

$$\begin{array}{rcl} \mathcal{A}(x) & = & \begin{pmatrix} a^{\star}_1(x) & 0 \\ 0 & a^{\star}_2(x) \end{pmatrix} + (a^{\star}_1(x) - a^{\star}_2(x))\ \sin\gamma \begin{pmatrix} -\sin\gamma & \cos\gamma \\ \cos\gamma & \sin\gamma \end{pmatrix}, \\ \underline{b}^0_1(x) & = & -\Big(b^{\star}_1(x)\ \Theta_1(x) - b^{\star}_2(x)\ \Theta_2(x)\Big)\ \sin^2\gamma + b^{\star}_1(x)\ \Theta_1(x), \\ \underline{b}^0_2(x) & = & \Big(b^{\star}_1(x)\ \Theta_1(x) - b^{\star}_2(x)\ \Theta_2(x)\Big)\ \sin\gamma\ \cos\gamma, \end{array}$$

and the following homogenized coefficients $(i, j = 1, 2$ and $j \neq i)$:

$$a^{\star}_i(x) = \frac{\widetilde{h^3_j}}{\widetilde{h^{-3}_i}}(x) \quad \textit{and} \quad b^{\star}_i(x) = \left[\frac{\widetilde{h^{-2}_i}}{\widetilde{h^{-3}_i}}\ \widetilde{h_j}\right](x).$$

Moreover, problem $(\mathcal{P}_\theta^\star)$ *admits* $(p_0,\ \Theta_1,\ \Theta_2)$ *as a solution, where*

$$\Theta_i(x) = \Big[\frac{1}{\widetilde{h_i^{-2}}\widetilde{h_j}}\ \widetilde{\Big(\frac{\theta_0 h_j}{h_i^2}\Big)}\Big](x), \quad i,j=1,2, \quad j \neq i, \tag{1.32}$$

and (p_0, θ_0) *is the two-scale limit of* $(p_\varepsilon, \theta_\varepsilon)$ *(solution of problem* $(\mathcal{P}_\theta^\varepsilon)$*).*

Proof. Theorem 1.36 is obtained from Theorem 1.34 using the inverse change of coordinates, with $\breve{f}(X,y) = f(x,y)$. □

Remark 1.37. *Theorem 1.36 implies that we have been able to solve one of the difficulties that raised in the most general case (see Remark 1.17). Indeed there are two saturation functions, but we have proved that they satisfy:* $0 \leq \Theta_i \leq 1$ *(*$i = 1,2$*), which was not guaranteed in the general case. In this way, the homogenized problem has a structure that is close to the initial one. But, as in the most general case, we cannot prove a uniqueness result with the methods of Section 1.1, nor can we numerically solve the problem using algorithms that have been previously mentioned, since we still have two saturation functions.*

Remark 1.38. *Let us recall that, in Theorem 1.16, we wrote the right-hand side as* $\underline{b}_i^0 = \Theta_i b_i^\star$*, thus defining "fake" saturation functions (since we were not able to prove that* $0 \leq \Theta_i \leq 1$*). In fact, according to Theorem 1.36,* $\underline{b}_i^0$ *should be considered as a combination of* $\Theta_i b_i^\star$*, where* Θ_i *can be considered as "real" saturation functions (since they satisfy* $0 \leq \Theta_i \leq 1$*).*

Remark 1.39. *Theorem 1.36 gives an example of an homogenized problem with non-diagonal terms in the matrix and additional homogenized coefficients in the second member (see Theorem 1.16 corresponding to the most general case). Indeed, let us try to understand the homogenized problem* $(\mathcal{P}_\theta^\varepsilon)$ *in a form that is a perturbation of the homogenized one defined under Assumption 4. For this, we define the main term* A^m *and the residual term* A^r *in the matrix as follows:*

$$\mathcal{A}(x) = \underbrace{\begin{pmatrix} a_1^\star(x) & 0 \\ 0 & a_2^\star(x) \end{pmatrix}}_{A^m(x)} + \underbrace{(a_1^\star(x) - a_2^\star(x))\ \sin\gamma \begin{pmatrix} -\sin\gamma & \cos\gamma \\ \cos\gamma & \sin\gamma \end{pmatrix}}_{A^r(x)}.$$

In the same way, we introduce in the second member the main component b_1^m *and the* *residual ones b_1^r and b_2^r:*

$$
\begin{aligned}
b_1^0(x) &= \underbrace{-\Big(b_1^\star(x)\ \Theta_1(x) - b_2^\star(x)\ \Theta_2(x)\Big)\ \sin^2\gamma}_{b_1^r} \qquad + \underbrace{\Theta_1(x)\ b_1^\star(x)}_{b_1^m},\\
b_2^0(x) &= \underbrace{\Big(b_1^\star(x)\ \Theta_1(x) - b_2^\star(x)\ \Theta_2(x)\Big)\ \sin\gamma\ \cos\gamma}_{b_2^r}.
\end{aligned}
$$

Let us notice that the main term in the second member only appears in the x_1 direction, corresponding to the flow direction. Moreover neglecting the residual terms in the formulation gives us the classical homogenized problem with $\gamma = k\pi/2$ ($k \in \mathbb{Z}$) (see Theorem 1.25).

Remark 1.40. *Considering the dam problem, an homogenized problem analogous to the initial one cannot be obtained in the most general case, since it is possible to show (see [Alt79, Mur03, Rod84]) that there exists the possibility of the non-convergence of the unsatured regions (i.e. $\{p_\varepsilon = 0\} \cap \{0 < \theta_\varepsilon < 1\}$). But the counter-example developped in the previous references is valid only for initial anisotropic permeability cases. In the lubrication case, this assumption is not relevant and the possibility to state an homogenized problem whose structure is similar to the initial one remains an open question.*

1.3 Numerical methods and results

In this section, the numerical simulation of a microhydrodynamic contact is performed to illustrate the theoretical results of convergence stated in the previous sections. For the numerical solution of the ε dependent problems and their corresponding homogenized one, we propose the characteristics method adapted to steady-state problems to deal with the convection term combined with a finite element spatial discretization. Moreover, the maximal monotone nonlinearity associated to the Elrod-Adams model for cavitation is treated by a duality method. The combination of these numerical techniques has been already successfully used in previous papers dealing with hydrodynamic aspects (see [BCV98, BD87]), and even with elastohydrodynamic aspects (see, for instance, [ACV02, DGV02]).

1.3.1 The characteristics method

■ *1st step - Time discretization* - Considering problem $(\mathcal{P}_\theta)$, the departure point is to introduce an artificial dependence on time t in the stationary functions, i.e. $\overline{\psi}(x,t) = \psi(x)$.

Considering the velocity field $\vec{u} = (-1, 0)$ and the corresponding total derivative operator,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla = -\frac{\partial}{\partial x_1},$$

then the stationary problem $(\mathcal{P}_\theta)$ gives place to the artificial evolutive one

$$\int_\Omega \theta h \, \frac{D\overline{\psi}}{Dt} + \int_\Omega h^3 \, \nabla p \, \nabla \overline{\psi} = 0 \quad \text{and} \quad \theta \in H(p).$$

Next, we consider the upwinded approximation of the total derivative

$$\frac{D\overline{\psi}}{Dt} \approx \frac{\psi(x) - \psi(X^k(x))}{k},$$

where k is an artificial time step and $X^k(x)$ denotes the position of a particle placed in the point x at time $t - k$ moving along the integral path of the velocity field $\vec{u}$, i.e. $X^k(x) = X(x, t; t - k)$. The function X is the solution of the O.D.E. problem for the characteristics

$$\frac{d}{d\tau} \left(X(x, t; \tau) \right) = u(X(x, t; \tau)) \quad \text{and} \quad X(x, t; t) = x.$$

In this way, the time discretized problem is written as

$$\int_\Omega \theta h \, \frac{\psi - \psi \circ X^k}{k} + \int_\Omega h^3 \, \nabla p \, \nabla \psi = 0 \quad \text{and} \quad \theta \in H(p),$$

which suggests to move the term containing $\psi \circ X^k$ into the right-hand side of the equation and to look for the solution of the evolutive problem as $t \to +\infty$.

■ *2nd step - Computation of one time step* - For each time step $t^n = n \; \Delta t$, the finite element discretization in space defines the final discretized problem

$$(\mathcal{P}_\Delta) \begin{cases} \dfrac{1}{k} \displaystyle\int_\Omega \theta_h^{n+1} h \; \psi_h + \int_\Omega h^3 \; \nabla p_h^{n+1} \; \nabla \psi_h = \frac{1}{k} \int_\Omega \theta_h^n h \; (\psi_h \circ X^k), \quad \forall \; \phi_h \in V_{oh}, \\ \theta_h^{n+1}(b) \in H(p^{n+1}(b)), \quad \forall \; b \text{ node of } \tau_h, \end{cases}$$

where τ_h is the triangularization of the domain. The finite element spaces are defined as

$$\begin{array}{rcl} V_h & = & \{ v_h \in C^0(\Omega), v_{h|E} \in P_1, \; \forall \; E \in \tau_h \}, \\ V_{oh} & = & \{ v_h \in V_h, \; v_{h|\Gamma_a \cup \Gamma_0} = 0 \}. \end{array}$$

Each iteration of the characteristics algorithm requires to solve the nonlinear problem

$(\mathcal{P}_\Delta)$. For this, we use the new unknown, r^{n+1}, defined by

$$r^{n+1} \in H(p^{n+1}) - \delta\, p^{n+1} \quad \text{in } \Omega,$$

δ being an arbitrary positive real constant. Then, dropping the subscripts h,

$$(\mathcal{P}_\Delta^\delta) \begin{cases} \dfrac{\delta}{k} \displaystyle\int_\Omega p^{n+1} h\, \psi \quad + \int_\Omega h^3\, \nabla p^{n+1}\, \nabla\psi \\ \qquad\qquad = \dfrac{1}{k} \displaystyle\int_\Omega \theta_h^n\, h\, (\psi \circ X^k) - \dfrac{1}{k} \int_\Omega r^{n+1}\, h\, \psi, \ \forall\, \phi_h \in V_{oh}, \\ r^{n+1} = H_\lambda^\delta(p^{n+1} + \lambda\, r^{n+1}), \end{cases}$$

where H_λ^δ denotes the Yosida approximation of $H - \delta I$, I being the identity operator. The fixed-point algorithm to solve $(\mathcal{P}_\Delta^\delta)$ proceeds as follows: at the beginning of each iteration we know r. Then we compute p as the solution of the linear problem $(\mathcal{P}_\Delta^\delta)$-(i) and update r with $(\mathcal{P}_\Delta^\delta)$-(ii).

1.3.2 Numerical tests

We adress the numerical simulation of journal bearing devices with circumferential supply of lubricant (see FIG.3, page 5). Indeed we simulate a journal-bearing device whose length is $L = 0.075\ m$, mean radius $R_m = (R_b + R_j)/2 = 0.0375\ m$ (R_b and R_j being the bearing and journal radii, respectively) and the clearance is $c = R_b - R_j = 0.001\ m$. The supply pressure is $p_a = 100000\ Pa$ or $p_a = 150000\ Pa$ (according to the case study), the lubricant viscosity is $\mu = 0.03382\ Pa.s$ and the velocity of the journal is taken to $v_0 = 30\ m/s$. Moreover, the smooth gap between the two surfaces is given by:

$$h(x) = c\left(1 + \rho\ \cos\left(\frac{x_1}{R_m}\right)\right), \qquad x = (x_1, x_2) \in]0, 2\pi R_m[\times]0, \frac{L}{2}[,$$

where the eccentricity ρ varies from 0.6 to 0.8 (according to the case study). The classical Reynolds problem, in real variables, should be posed as:

$$\begin{cases} \nabla \cdot \Big(\dfrac{h_s^3}{6\mu} \nabla p\Big) = v_0 \dfrac{\partial}{\partial x_1}\Big(\theta h_s\Big), & in\ ]0, 2\pi R_m[\times]0, L/2[, \\ p \geq 0, \quad p\,(1-\theta) = 0, \quad 0 \leq \theta \leq 1, & in\ ]0, 2\pi R_m[\times]0, L/2[, \end{cases}$$

with the boundary conditions:

$$p = 0, \text{ on }]0, 2\pi R_m[\times\{0\} \text{ and } p = p_a, \text{ on }]0, 2\pi R_m[\times\{L/2\},$$
$$p \text{ and } v_0\ \theta h_s - \frac{h_s^3}{6\mu}\frac{\partial p}{\partial x_1} \text{ are } 2\pi R_m x_1 \text{ periodic.}$$

Now let us introduce the dimensionless coordinates and quantities that provide the effective system to solve (see [ACV02]):

$$X_1 = \frac{x_1}{R_m}, \qquad X_2 = \frac{2\ y_2}{L}, \qquad H_s(X) = \frac{h_s(x)}{c},$$
$$P = \frac{c^2}{6v_0 R_m \mu}\ p, \quad P_a = \frac{c^2}{6v_0 R_m \mu}\ p_a, \qquad \kappa = \frac{2R_m}{L}.$$

Then, the dimensionless Reynolds problem becomes:

$$\begin{cases} \dfrac{\partial}{\partial X_1}\Big(H_s^3 \dfrac{\partial P}{\partial X_1}\Big) + \kappa^2 \dfrac{\partial}{\partial X_2}\Big(H_s^3 \dfrac{\partial P}{\partial X_2}\Big) = \dfrac{\partial}{\partial X_1}\Big(\theta H_s\Big), & in\ ]0,2\pi[\times]0,1[, \\ P \geq 0, \quad P\ (1-\theta) = 0, \quad 0 \leq \theta \leq 1, & in\ ]0,2\pi[\times]0,1[, \end{cases}$$

with the boundary conditions:

$$P = 0, \text{ on }]0,2\pi[\times\{0\} \text{ and } P = P_a, \text{ on }]0,2\pi[\times\{1\},$$
$$P \text{ and } \theta H_s - H_s^3 \frac{\partial P}{\partial X_1} \text{ are } 2\pi X_1 \text{ periodic,}$$

and the smooth gap is now $H_s(X) = 1 + \rho\ \cos(X_1)$. Let us now introduce the roughness patterns: we propose in the rough case the following expression for the dimensionless gap:

$$H_\varepsilon(X) = H\left(X, \frac{X}{\varepsilon}\right) = \begin{cases} H_s(X) + h_r \sin\left(\dfrac{X_1}{\varepsilon}\right), & \text{for transverse roughness,} \\ H_s(X) + h_r \sin\left(2\pi \dfrac{X_2}{\varepsilon}\right), & \text{for longitudinal roughness,} \end{cases}$$

where h_r denotes the amplitude of the roughnesses and ε represents the spacing of the roughness. In order to guarantee the positivity of the gap, we choose h_r so that $h_r > 1-\rho$. The homogenized problem to solve can be written under the form:

$$\begin{cases} \dfrac{\partial}{\partial X_1}\Big(a_1 \dfrac{\partial P_0}{\partial X_1}\Big) + \kappa^2 \dfrac{\partial}{\partial X_2}\Big(a_2 \dfrac{\partial P_0}{\partial X_2}\Big) = \dfrac{\partial}{\partial X_1}\Big(\Theta b\Big), & in\ ]0,2\pi[\times]0,1[, \\ P \geq 0, \quad P_0\ (1-\Theta) = 0, \quad 0 \leq \Theta \leq 1, & in\ ]0,2\pi[\times]0,1[, \end{cases}$$

with the boundary conditions:

$$P_0 = 0, \text{ on }]0,2\pi[\times\{0\} \text{ and } P_0 = P_a, \text{ on }]0,2\pi[\times\{1\},$$
$$P_0 \text{ and } \Theta b - a_1 \frac{\partial P_0}{\partial X_1} \text{ are } 2\pi X_1 \text{ periodic.}$$

In TABLE 1.1, we present the coefficients a_1, a_2 and b that appear in the homogenized problem for purely transverse and purely longitudinal roughness cases which have been computed with MATHEMATICA Software Package:

	Transverse roughness	Longitudinal roughness
$H(X,Y)$	$H_s(X) + h_r \sin(Y_1)$	$H_s(X) + h_r \sin(2\pi Y_2)$
$a_1(X)$	$2\dfrac{\left(H_s(X)^2 - h_r^2\right)^{5/2}}{2H_s(X)^2 + h_r^2}$	$H_s(X)^3 + \dfrac{3}{2}\ H_s(X)\ h_r^2$
$a_2(X)$	$H_s(X)^3 + \dfrac{3}{2}\ H_s(X)\ h_r^2$	$2\ \dfrac{\left(H_s(X)^2 - h_r^2\right)^{5/2}}{2H_s(X)^2 + h_r^2}$
$b(X)$	$2H_s(X)\ \dfrac{H_s(X)^2 - h_r^2}{2H_s(X)^2 + h_r^2}$	$H_s(X)$

Table 1.1. Hydrodynamic homogenized coefficients

Case 1: Transverse roughness tests

Numerical tests have been made for two different regimes: the first one is a realistic regime in terms of the size of the roughness linked to mechanical applications; the second one is a severe unrealistic regime, since the deformability of the surface should be taken into account. However, in both cases, we have considered the following physical data: the eccentricity is $\rho = 0.6$. The numerical methods parameters are the following ones: a triangular uniform finite element mesh whose parameters Δx_1 and Δx_2 are given further, an artificial time step for the characteristics method (see [BCV98]) $\Delta t = \Delta x_1$; the Bermúdez-Moreno parameters are $\omega = 1$ and $\lambda = 1/(2\omega)$; the stopping test in all algorithms is equal to $\delta = 10^{-4}$ (corresponding to the absolute error in the discrete L^∞ norm between two iterations in time).

■ *Case* 1^a*:* The amplitude of the roughness is given by $\beta/(1-\rho) = 0.5$. The mesh parameters are $\Delta x_1 = 2\pi/600$ and $\Delta x_2 = 1/50$, so that we have 60000 triangles and 30651 vertices. Numerical tests illustrate the two-scale convergence results established in previous sections. In particular, FIG.1.7 and 1.8 represent the cuts at $x_2 = 0.0016\ m$ for the pressure and saturation variables for different numbers of roughness patterns $N_\varepsilon = 2\pi/\varepsilon$ and the homogenized solution. The figures illustrate the convergence of the pressure but also the behaviour of the cavitation function:

- FIG.1.7: it illustrates the strong convergence of p_ε to p_0 in $L^2(\Omega)$.
- FIG.1.8: as pointed out in Remark 1.20, it is clear that θ_ε converges in $L^2(\Omega)$ only in a weak sense; in particular, one sees that the amplitude of the gradient explodes when $\varepsilon \to 0$, so that $\theta_0(x, y)$ actually depends on the y variable.

Finally, FIG.1.9 and 1.10 present the homogenized pressure and saturation in the whole domain.

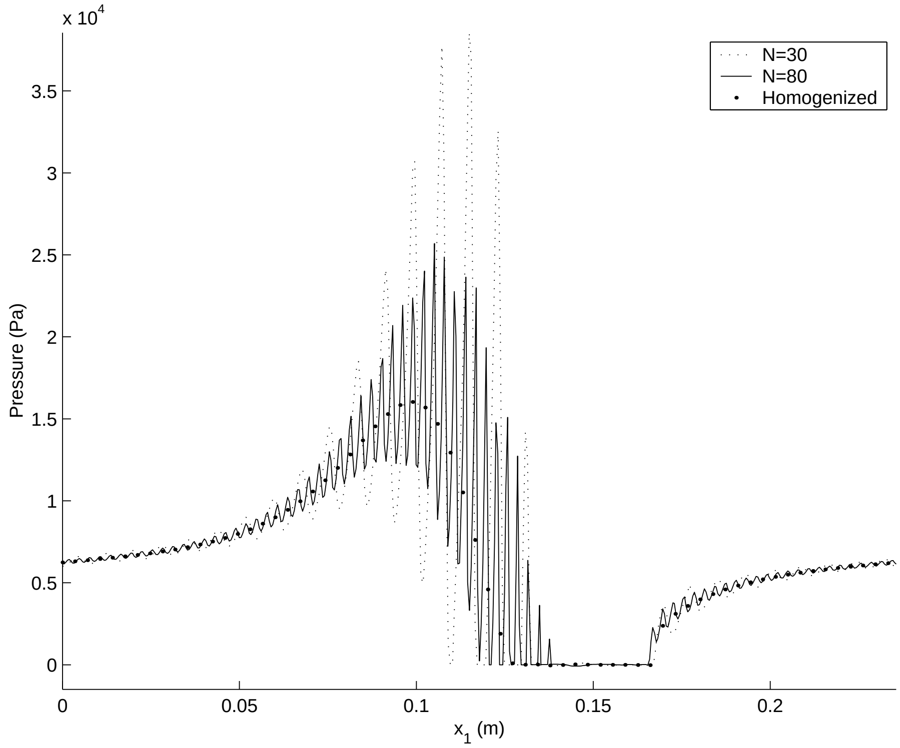


Figure 1.7. Hydrodynamic pressure at $x_2 = 0.0016\ m$ (transverse roughness; case 1^a)

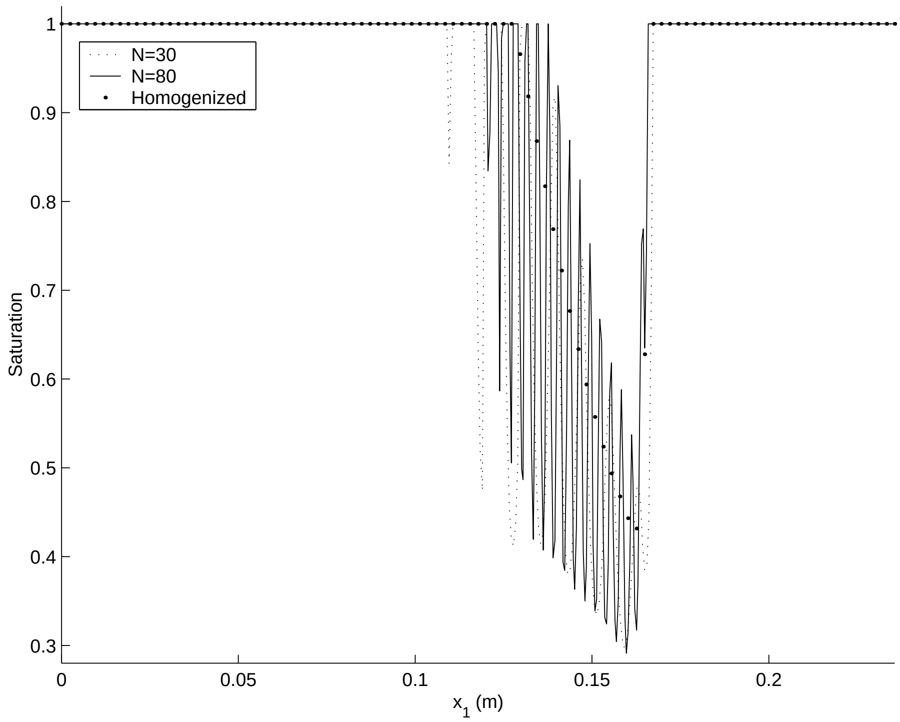


Figure 1.8. Hydrodynamic saturation at $x_2 = 0.0016\ m$ (transverse roughness; case 1^a)

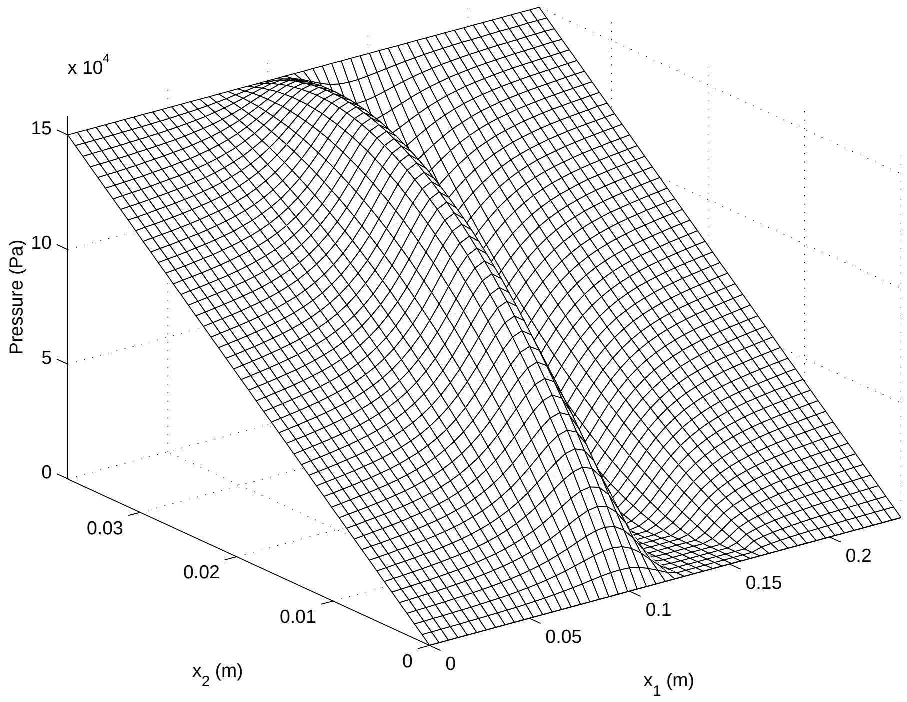


Figure 1.9. Hydrodynamic homogenized pressure (transverse roughness; case 1^a)

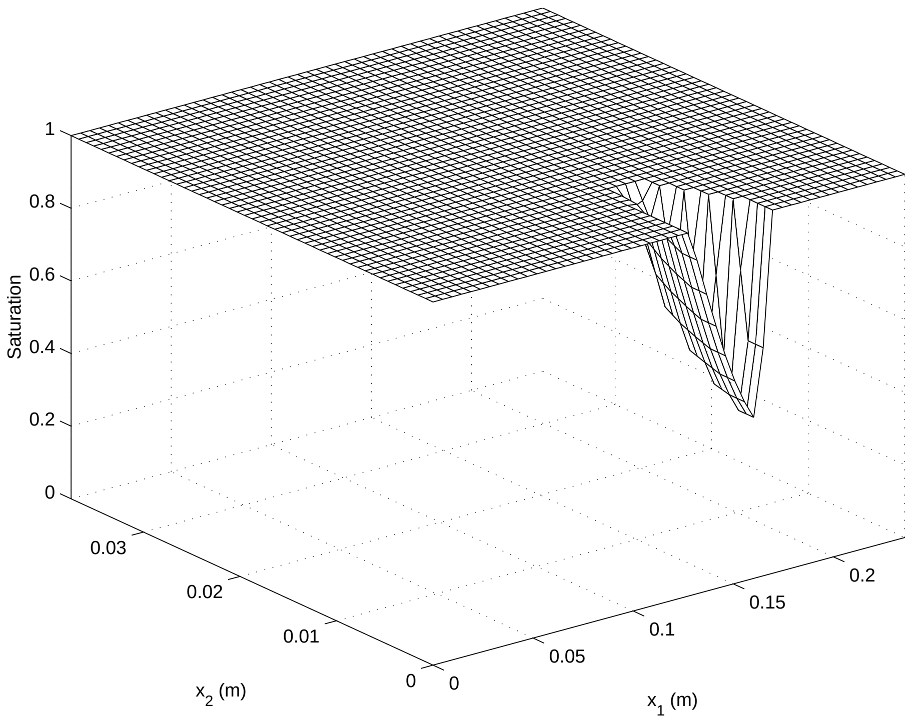


Figure 1.10. Hydrodynamic homogenized saturation (transverse roughness; case 1^a)

■ *Case* 1^b*:* In this severe regime, the amplitude of the roughness is given by $\beta/(1-\rho) = 0.9$. The mesh parameters are $\Delta x_1 = 2\pi/400$ and $\Delta x_2 = 1/50$, so that we have 40000 triangles and 20451 vertices. FIG.1.12 and 1.13 represent the cuts at $x_2 = 0.0032\ m$ for the pressure and saturation variables for different numbers of roughness patterns $N_\varepsilon = 2\pi/\varepsilon$ and the homogenized solution.

FIG.1.12 and 1.13 illustrate the convergence results. The comments that have been established in Case 1^a are still valid, even in a severe regime. Let us notice that numerical computations become very difficult when N_ε becomes greater than 60: it is, of course, a case which really falls into the scope of homogenization studies and shows the interest of the method.

Finally, let us denote r_{N_ε} the residual term

$$r_{N_\varepsilon} = \Big\| p_\varepsilon - p_0 \Big\|_{L^2(\Omega)}.$$

Supposing that p_ε converges strongly to p_0 in $L^2(\Omega)$ with an order of convergence $\mathcal{O}(\varepsilon^\alpha)$, we numerically calculate α: FIG.1.11 is obtained so that $\alpha = 1$.

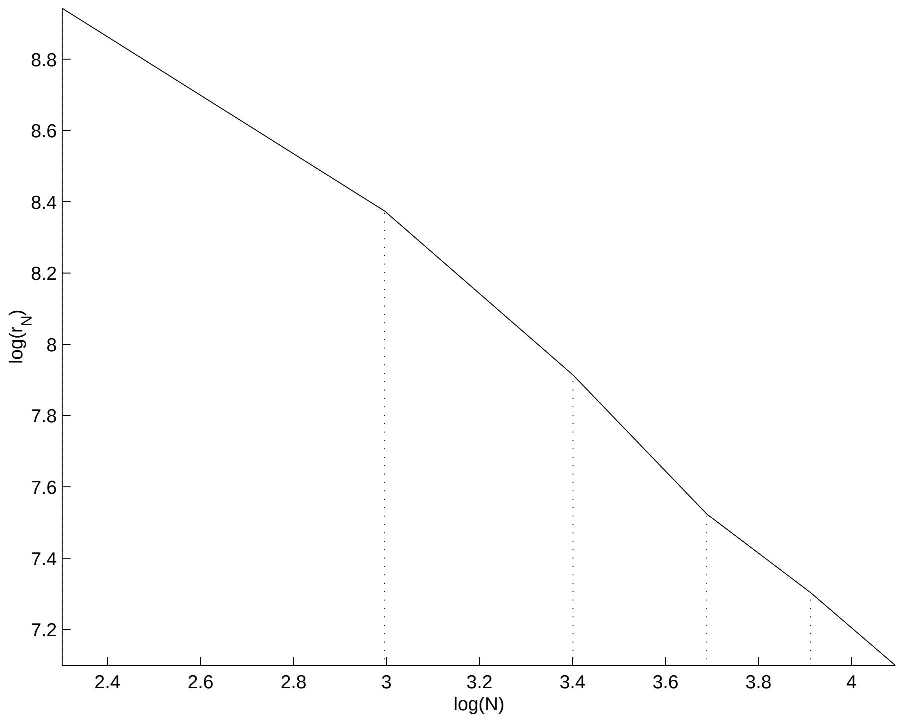


Figure 1.11. Convergence speed of the pressure (transverse roughness; case 1^b)

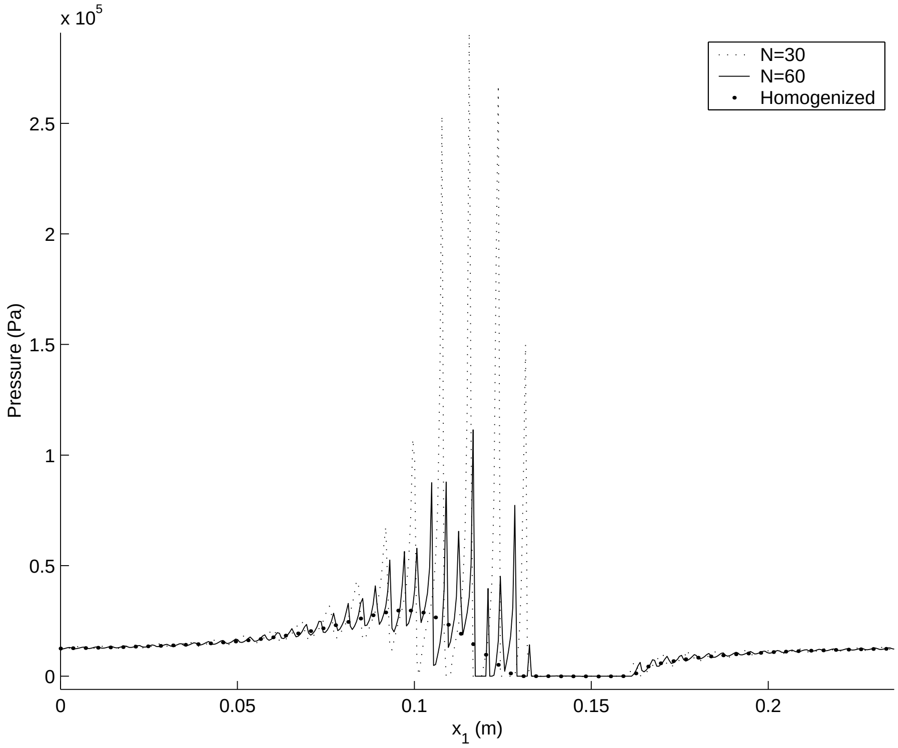


Figure 1.12. Hydrodynamic pressure at $x_2 = 0.0032\ m$ (transverse roughness; case 1^b)

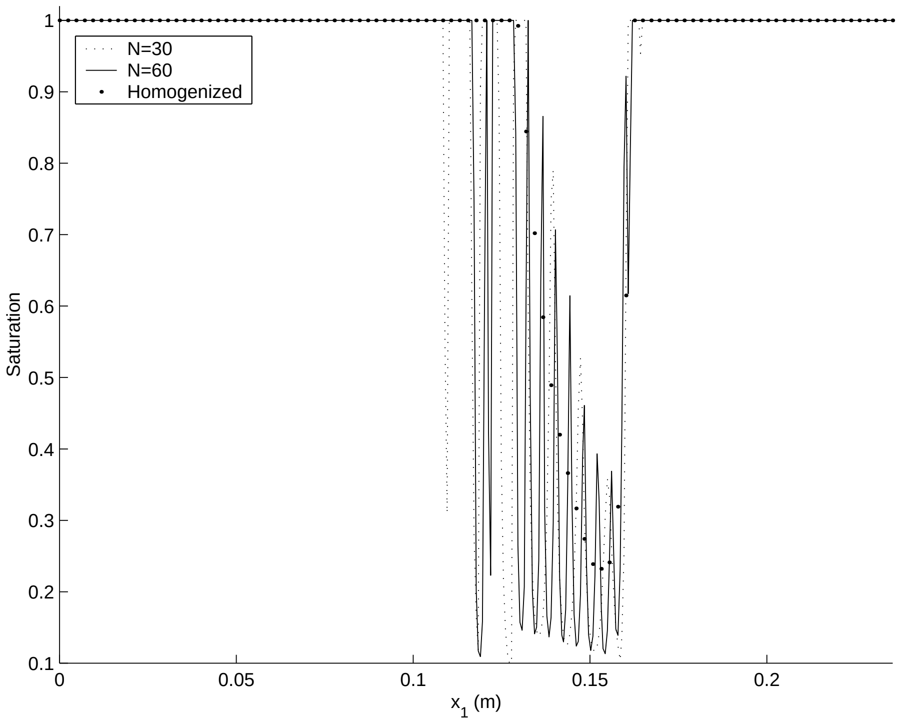


Figure 1.13. Hydrodynamic saturation at $x_2 = 0.0032\ m$ (transverse roughness; case 1^b)

Case 2: Longitudinal roughness tests

For this test, we have considered the following physical data: the eccentricity is $\rho = 0.8$ and the amplitude of the roughness is given by $\beta/(1-\rho) = 0.5$, which is a realistic regime in terms of mechanical applications. The numerical methods parameters are the following ones: a triangular uniform finite element mesh with $\Delta x_1 = 2\pi/400$, $\Delta x_2 = 1/80$ (so that we have 64000 triangles and 32481 vertices), an artificial time step for the characteristics method $\Delta t = \Delta x_1$; the Bermúdez-Moreno parameters are still $\omega = 1$ and $\lambda = 1/(2\omega)$; the stopping test in all algorithms is equal to $\delta = 10^{-5}$.

FIG.1.14 and 1.15 represent the cuts at $x_1 = 0.1060\ m$ and $x_1 = 0.1355\ m$ respectively, for the deterministic pressure (for different numbers of roughness patterns $N_\varepsilon = 2\pi/\varepsilon$) and the homogenized pressure.

- FIG.1.14: the section of the bearing does not contain any cavitation area ($p > 0$) so that the saturation function is constant and equal to 1 (therefore the corresponding figure is omitted). Notice that the section corresponds to the minimum gap (and maximum pressure).
- FIG.1.15: in this case, the section does contain a cavitated area.

Thus, the figures allow us to observe convergence phenomena for the pressure in both cavitated and non-cavitated areas. Let us notice that, not surprisingly, the convergence of the pressure is better in the longitudinal roughness case, as the influence of the roughness on the pressure is relatively small. As in the transverse roughness tests, we could numerically illustrate the weak convergence of the saturation. Finally, FIG.1.16 and 1.17 present the homogenized pressure and saturation in the whole domain.

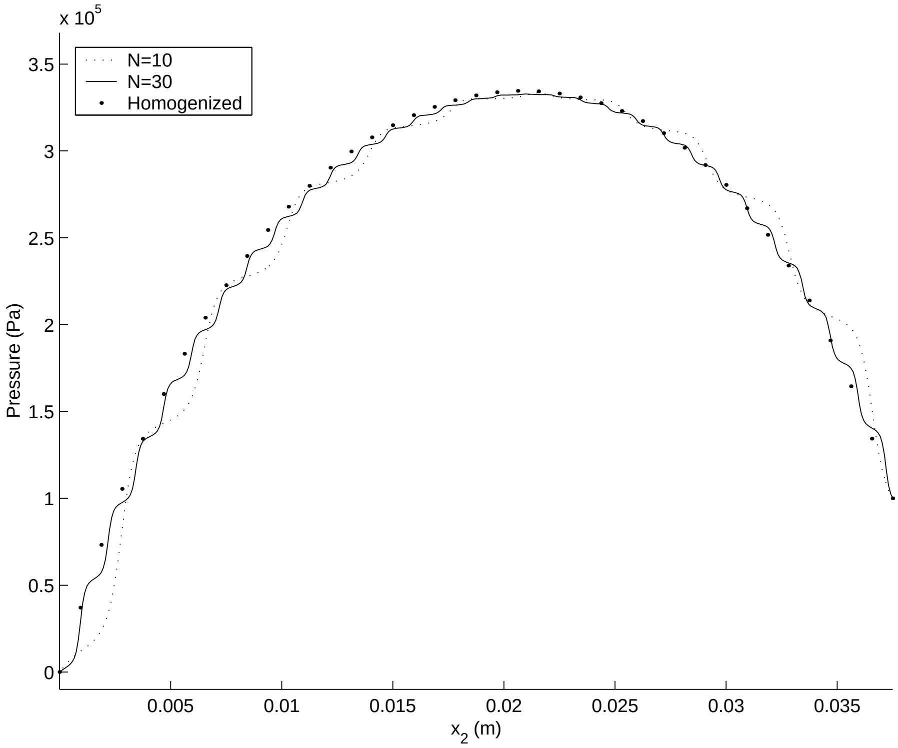


Figure 1.14. Hydrodynamic pressure at $x_1 = 0.1060\ m$ (longitudinal roughness; case 2)

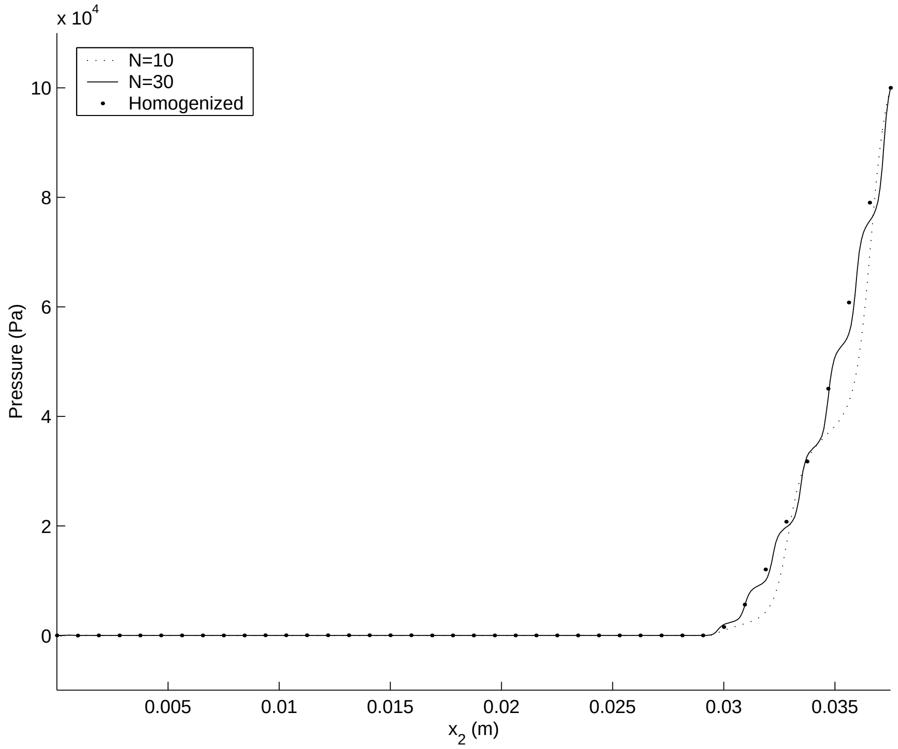


Figure 1.15. Hydrodynamic pressure at $x_1 = 0.1355\ m$ (longitudinal roughness; case 2)

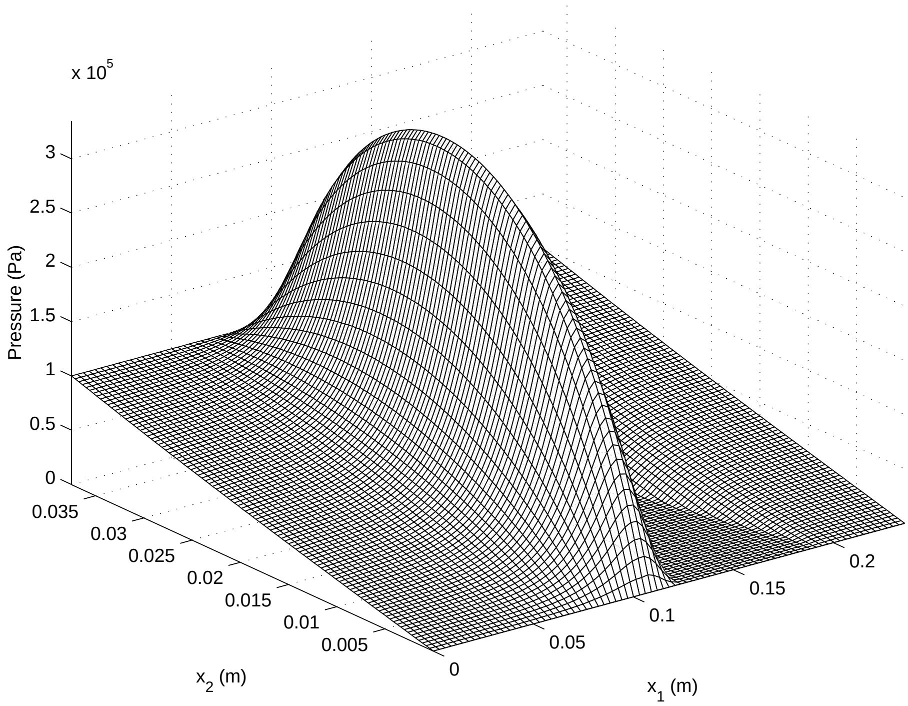


Figure 1.16. Hydrodynamic homogenized pressure (longitudinal roughness; case 2)

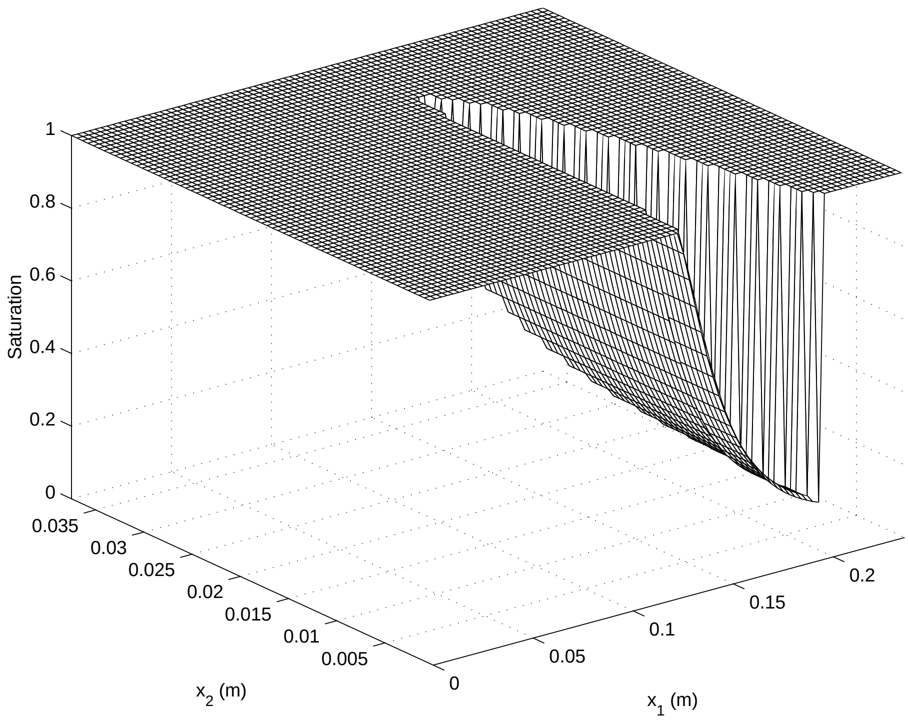


Figure 1.17. Hydrodynamic homogenized saturation (longitudinal roughness; case 2)

Bibliography

[ACV02] I. Arregui, J. J. Cendán, and C. Vázquez. Mathematical analysis and numerical simulation of a Reynolds-Koiter model for the elastohydrodynamic journal-bearing device. *M2AN Math. Model. Numer. Anal.*, 36(2):325–343, 2002.

[All92] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.

[Alt79] H. W. Alt. Strömungen durch inhomogene poröse Medien mit freiem Rand. *J. Reine Angew. Math.*, 305:89–115, 1979.

[Alt81] H. W. Alt. Numerical solution of steady-state porous flow free boundary problems. *Numer. Math.*, 36(1):73–98, 1980/81.

[Alv86] S. J. Alvarez. Problemas de frontera libre en teoria de lubrificacion. *Tesis de doctorado, Universidad Complutense de Madrid, Spain*, 1986.

[AO03] S. J. Alvarez and R. Oujja. On the uniqueness of the solution of an evolution free boundary problem in theory of lubrication. *Nonlinear Anal.*, 54(5):845–872, 2003.

[Bai80] C. Baiocchi. Free boundary problems in fluid flow through porous media and variational inequalities. in *Free boundary problems, Vol. I (Pavia, 1979)*, pages 175–191. Ist. Naz. Alta Mat. Francesco Severi, Rome, 1980.

[BB93] G. Bayada and M. Boukrouche. Mathematical model. Existence and uniqueness of cavitation problems in porous journal bearing. *Nonlinear Anal.*, 20(8):895–920, 1993.

[BC86] G. Bayada and M. Chambat. The transition between the Stokes equations and the Reynolds equation: a mathematical proof. *Appl. Math. Optim.*, 14(1):73–93, 1986.

[BC88] G. Bayada and M. Chambat. New models in the theory of the hydrodynamic lubrication of rough surfaces. *ASME J. Tribol.*, 110:402–407, 1988.

[BC89] G. Bayada and M. Chambat. Homogenization of the Stokes system in a thin film flow with rapidly varying thickness. *RAIRO Modél. Math. Anal. Numér.*, 23(2):205–234, 1989.

[BCF88] G. Bayada, M. Chambat, and J.-B Faure. Some effects of the boundary roughness in a thin film flow. in *Boundary Control and Boundary Variations, Lecture Notes in Control and Information Science, Vol. 100*, pages 96–115. Springer-Verlag, (1988).

[BCMP73] C. Baiocchi, V. Comincioli, E. Magenes, and G.A. Pozzi. Free boundary problems in the theory of fluid flow through porous media: existence and uniqueness theorems. *Ann. Mat. Pura Appl. (4)*, 97:1–82, 1973.

[BCV98] G. Bayada, M. Chambat, and C. Vázquez. Characteristics method for the formulation and computation of a free boundary cavitation problem. *J. Comput. Appl. Math.*, 98(2):191–212, 1998.

[BD87] A. Bermúdez and J. Durany. La méthode des caractéristiques pour les problèmes de convection-diffusion stationnaires. *RAIRO Modél. Math. Anal. Numér.*, 21(1):7–26, 1987.

[Ben74] V. Benci. On a filtration problem through a porous medium. *Ann. Mat. Pura Appl. (4)*, 100:191–209, 1974.

[BF77] C. Baiocchi and A. Friedman. A filtration problem in a porous medium with variable permeability. *Ann. Mat. Pura Appl. (4)*, 114:377–393, 1977.

[BF89] G. Bayada and J.-B Faure. A double-scale analysis approach of the Reynolds roughness. Comments and application to the journal bearing. *ASME J. Tribol.*, 111:323–330, 1989.

[BKS78] H. Brezis, D. Kinderlehrer, and G. Stampacchia. Sur une nouvelle formulation du problème de l'écoulement à travers une digue. *C. R. Acad. Sci. Paris Sér. A-B*, 287(9):A711–A714, 1978.

[BLP78] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structures*, volume 5 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1978.

[Bre83] H. Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[CC83] G. Capriz and G. Cimatti. Partial lubrication of full cylindrical bearings. *ASME J. Lubrication Technol.*, 105:84–89, 1983.

[CDG02] D. Cioranescu, A. Damlamian, and G. Griso. Periodic unfolding and homogenization. *C. R. Math. Acad. Sci. Paris*, 335(1):99–104, 2002.

[CE70] J. Coyne and H. G. Elrod. Conditions for the rupture of a lubricating film, Part 1. *ASME J. Lubrication Technol.*, 92:451–456, 1970.

[CE71] J. Coyne and H. G. Elrod. Conditions for the rupture of a lubricating film, Part 2. *ASME J. Lubrication Technol.*, 93:156–167, 1971.

[Dar56] H. Darcy. *Les fontaines publiques de la ville de Dijon*. Dalmont, Paris, first edition, 1856.

[DGV02] J. Durany, G. García, and C. Vázquez. Numerical simulation of a lubricated Hertzian contact problem under imposed load. *Finite Elem. Anal. Des.*, 38(7):645–658, 2002.

[EA75] H. G. Elrod and M. L. Adams. A computer program for cavitation. *Cavitation and related phenomena in lubrication - Proceedings - Mech. Eng. Publ. ltd*, pages 37–42, 1975.

[GT01] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[HS01] S. H. Harp and R. F. Salant. An average flow model of rough surface lubrication with inter-asperity cavitation. *ASME J. Tribol.*, 123:134–143, 2001.

[LNW02] D. Lukkassen, G. Nguetseng, and P. Wall. Two-scale convergence. *Int. J. Pure Appl. Math.*, 2(1):35–86, 2002.

[MP86] L. D. Marini and P. Pietra. Fixed-point algorithms for stationary flow in porous media. *Comput. Methods Appl. Mech. Engrg.*, 56(1):17–45, 1986.

[Mur03] F. Murat. Personal communication, 2003.

[Ngu89] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20(3):608–623, 1989.

[PC78] N. Patir and H. S. Cheng. An average flow model for determining effects of three-dimensional roughness on partial hydrodynamic lubrication. *ASME J. Lubrication Technol.*, 100:12–17, 1978.

[PC79] N. Patir and H. S. Cheng. Application of average flow model to lubrication between rough sliding surfaces. *ASME J. Lubrication Technol.*, 101:220–230, 1979.

[Rey86] O. Reynolds. On the theory of lubrication and its application to Mr Beauchamp tower's experiments, including an experimental determination of the viscosity of olive oil. *Phil. Trans. Roy. Soc.*, A 117:157–234, 1886.

[Rey99] O. Reynolds. On the slipperiness of ice. *Man. Lit. Phil. Soc., Memoirs and Proceedings*, 43:199–220, 1898-99.

[Rod84] J.-F. Rodrigues. Some remarks on the homogenization of the dam problem. *Manuscripta Math.*, 46(1-3):65–82, 1984.

[SS00] F. Shi and R. F. Salant. A mixed soft elastohydrodynamic lubrication model with interasperity cavitation and surface shear deformation. *ASME J. Tribol.*, 122:308–316, 2000.

[Tri83] J. H. Tripp. Surface roughness effects in hydrodynamic lubrication: the flow factor method. *ASME J. Lubrication Technol.*, 105:458–465, 1983.

[VOV03] D. E. A. Van Odyck and C. H. Venner. Compressible Stokes flow in thin films. *ASME J. Tribol.*, 125:543–551, 2003.

2

Homogenization of a nonlocal elastohydrodynamic lubrication problem

Article paru dans
Mathematical Models and Methods in Applied Sciences

Abstract *The present chapter deals with the homogenization of a lubrication problem, using the periodic unfolding method. We study in particular the Reynolds-Hertz model, which takes into account elastohydrodynamic deformations of the upper surface, when highly oscillating roughness effects occur. The difficulty arises when considering cavitation free boundary phenomena, leading to highly nonlinear and nonlocal problems.*

2.0 Statement of the problem

The greatly increasing number of industrial technical devices involving the presence of lubricated contacts motivates interest in studying more suitable models for the practical situations. Considering an elastic rolling ball or cylinder and a rigid plane leads to an elastohydrodynamic lubrication problem, taking into account the possibility of the ball/cylinder deformation. From a practical point of view, the introduction of surface periodic roughness during manufacturing processes cannot be avoided. In this rough elastohydrodynamic contact setting, it is important to state adequate mathematical models in order to perform the numerical simulation of the devices. For this, one possible tool is provided by the homogenization technique analyzed in the present chapter.

The mathematical model, to be further detailed later, is governed by the following set of highly coupled and nonlinear equations:

$$\nabla \cdot \left(h[p]^3 \; e^{-\alpha p} \; \nabla p\right) = \frac{\partial}{\partial x_1} \left(\theta h[p]\right)$$
$$p \geq 0, \quad 0 \leq \theta \leq 1, \quad p(1-\theta) = 0,$$

where $h[p]$, which is the effective gap between two close surfaces, contains a given rigid contribution h_r and an elastic one, which strongly depends on the main unknown p (lubricant pressure) in the following nonlocal form:

$$h[p] = h_r + \int_{\Omega} k(\cdot, z) p(z) \; dz,$$

the kernel k depending on the kind of contact. Moreover, the fluid saturation, θ, is related to the pressure by means of the Heaviside multivalued operator H and the exponential term takes into account piezoviscous effects.

Basic aspects of the early developped elastohydrodynamic theory have been stated by Dowson and Higginson [DH77], where the three main common features of this kind of problems are already quoted: the fluid hydrodynamic displacement, the solid elastic deformation and the air bubble generation. Thus, the Reynolds equation, linear Hertz contact theory and different cavitation models try to mathematically analyse these three phenomena. Moreover, the modification of the initial fluid viscosity due to the presence of sufficiently high values of lubricant pressure might have to be taken into account, so that the complete modelling is extended to piezoviscous fluids. Thus the complete model takes into account the following aspects:

- The Reynolds equation has been used for a long time to describe the behaviour of a viscous flow between two close surfaces in relative motion [Rey86]. It can be written as:
$$\nabla \cdot \left(\frac{h_r^3}{6\mu} \nabla p\right) = v_0 \frac{\partial}{\partial x_1} \left(h_r\right)$$
where p is the pressure distribution, h_r the gap between the two surfaces, μ the lubricant viscosity and v_0 the speed of the lower surface (the upper surface is assumed to be fixed). The transition of the Stokes equation to the Reynolds equation has been proved by Bayada and Chambat in [BC86b].

- However, the earlier equation does not take into account cavitation phenomena: cavitation is defined as the rupture of the continuous film due to the formation of air bubbles and makes the Reynolds equation no longer valid in the cavitation area. In order to make it possible, various models have been used, the most popular perhaps being variational inequalities which have a strong mathematical basis but lack physical evidence. Thus, we use the Elrod-Adams model, which introduces the hypothesis that the cavitation region is a fluid-air mixture and an additional unknown θ (the saturation of fluid in the mixture) (see [EA75, BC86a]).

- Actually, elastohydrodynamic lubrication (EHL) occurs between point or line contact, so all the loading is concentrated over a small contact area. Typical applications are rolling element bearings, most gears, and cams and tappets [DH77]. The concentrated contact results in high peak pressures of 1-2 GPa between the surfaces. This is too high to be supported by a normal hydrodynamic film, and application of simple hydrodynamic theory predicts negligible oil film thickness. In practice films are formed and have thickness comparable to the surface roughness of normal gear and bearing materials. This is because the high pressure has two beneficial effects unaccounted for in hydrodynamics. Firstly, elastic flattening of the contacting surfaces occurs. Secondly, the high pressure greatly increases the viscosity of the lubricant in the contact. Elastohydrodynamic lubrication is consequently analysed using a combination of Reynold's equation, elasticity theory (the Hertz equation) and a lubricant viscosity-pressure equation. Thus, introducing the elastic deformation of the surfaces due to the fluid pressure and assuming the Hertzian contact theory for a parameter regime that corresponds to low speeds, low viscosity at ambient pressure or small elastic modulus, the effective gap is, in fact, linked to the pressure. Let us mention that piezoviscous properties of the fluid have to be taken into account in realistic applications. Thus, the viscosity is no longer constant and also depends on the pressure [Bar93].

The mathematical analysis of different elastohydrodynamic problems taking into account the previous quoted features has been treated in the literature for the variational inequality cavitation model [GN94, Hu90, OW85, Rod93] and for the Elrod-Adams model [BEATV96, DGV96a].

The effect of surfaces periodic roughness on the behaviour of hydrodynamic and elastohydrodynamic magnitudes has been treated in numerous works. Some of the theoretical studies also include numerical examples which show how significant pressure and deformation perturbations appear due to the presence of surface asperities, either in the hydrodynamic regime [BF89, BMV05, Jai95] or in the elastohydrodynamic one [BCV03]. For the particular point and linear elastohydrodynamic contacts here treated, although some numerical methods have been proposed in the literature [Hoo98], the rigorous mathematical analysis to justify the homogenized models has not been performed yet. Furthermore, in this chapter, the more realistic Elrod-Adams model is considered instead of the variational inequality one [BCV03]. Notice that for the numerical simulation of micro-elastohydrodynamic contacts, the statement of well justified homogenized problems prevents from using extremely high numbers of mesh points to accurately compute the involved physical magnitudes (when directly solving the small parameter dependent problems).

Now, we present the full lubrication model, including cavitation phenomena and piezo-viscous elastohydrodynamic aspects.

2.1 Mathematical formulation of the EHL problem

We consider a rectangular domain $\Omega =]-l_1, l_1[\times]-l_2, l_2[$; $\Gamma_\star$ denotes the left vertical boundary and $\Gamma = \partial\Omega \backslash \Gamma_\star$ (see for instance FIG.6, page 6). We suppose that the following assumptions are satisfied.

Assumption 6 (Rigid contribution to the gap)**.** *The classical approximation of the rigid gap (see [DH77]) is given by the expression*

$$h_r(x) = \begin{cases} h_0 + \dfrac{x_1^2 + x_2^2}{2R}, & \text{for ball bearings} \\ h_0 + \dfrac{x_1^2}{R}, & \text{for linear bearings} \end{cases} \tag{2.1}$$

that represents a parabolic approximation for a given sphere-plane (point contact) or cylinder-plane (line contact) gap, R being the sphere or cylinder section radius.

Remark 2.1. *The positive constant h_0 corresponds to the gap at the point nearest to contact. Clearly, the condition*

$$0 < h_0 \leq h_r(x) \leq h_1, \quad \text{with } h_0,\, h_1 \text{ two constants} \tag{2.2}$$

is satisfied in the bounded domain Ω.

As previously mentioned, in the Hertz theory, the effective gap is linked to the pressure through the following relationship

Assumption 7 (Deformable contribution to the gap)**.** *The effective gap between the surfaces is given by*

$$h[p](x) = h_r(x) + \int_\Omega k(x,z)p(z)\ dz, \quad \forall\ x \in \Omega \tag{2.3}$$

where h_r satisfies Assumption 6, and $k(x,z)$ is

$$k(x,z) = \begin{cases} c_0\ \log\left|\dfrac{c_1 - z_1}{x_1 - z_1}\right|, & \text{for line contacts} \\ \dfrac{c_0}{\sqrt{(x_1-z_1)^2 + (x_2-z_2)^2}}, & \text{for point contacts,} \end{cases} \tag{2.4}$$

where $c_0 > 0$ *and* $c_1 \geq \max\{\left|x_1\right|,\ x \in \overline{\Omega}\}$.

Remark 2.2. *Clearly* k *is a positive function and there exists* $\tilde{K} > 0$ *such that*

$$\left\|k(x,.)\right\|_{L^1(\Omega)} \leq \tilde{K} \tag{2.5}$$

uniformly with respect to x. *Let us notice that the expression of* h *contains a rigid term,* h_r, *and an additional term due to the surface deformation.*

Finally let us take into account the piezoviscous properties of the lubricant, i.e. the viscosity is no longer constant.

Assumption 8 (Piezoviscosity law)**.** *The viscosity obeys the Barus law [Bar93]:*

$$\mu = \mu_0\ e^{\alpha p}, \tag{2.6}$$

where $\alpha \geq 0$ *and* $\mu_0 > 0$ *denote the piezoviscosity constant and the zero pressure viscosity repectively.*

The strong formulation of the problem is:

$$x \in \Omega^+ \quad : \quad \begin{cases} \nabla \cdot \left(\dfrac{h^3[p](x)}{6\mu_0}\ e^{-\alpha p(x)}\ \nabla p(x) \right) = v_0 \dfrac{\partial}{\partial x_1}\Big(\theta(x)\ h[p](x)\Big) \\ p(x) > 0 \quad \text{and} \quad \theta(x) = 1 \end{cases}$$

$$x \in \Omega_0 \quad : \quad \begin{cases} v_0\ \dfrac{\partial}{\partial x_1}\Big(\theta(x)\ h[p](x)\Big) = 0 \\ p(x) = 0 \quad \text{and} \quad 0 \leq \theta(x) \leq 1 \end{cases}$$

$$x \in \Sigma \quad : \quad \begin{cases} \dfrac{h^3[p](x)}{6\mu_0}\ e^{-\alpha p(x)}\ \dfrac{\partial p(x)}{\partial n} = v_0\ (1-\theta(x))h[p](x)\ \cos(\vec{n},\vec{i}) \\ p(x) = 0 \end{cases}$$

with the boundary conditions

$$\begin{aligned} & v_0\ \theta h[p] - \frac{h^3[p]}{6\mu_0}\ e^{-\alpha p}\ \frac{\partial p}{\partial n} = v_0\ \theta_\star h[p] && \text{on } \Gamma_\star \\ & p = 0 && \text{on } \Gamma \end{aligned}$$

where v_0 denotes the velocity of the lower surface in the x_1 direction, $\theta_\star$ is a supply parameter belonging to $[0, 1]$, $\vec{n}$ represents the unit normal vector to Σ pointing to Ω_0, $\vec{i}$

is the unit vector in the x_1 direction; and the sets appearing earlier are given by

$$\begin{array}{rcll}\Omega^+ & = & \{x \in \Omega,\ p(x) > 0\} & \text{(lubricated region)},\\ \Omega_0 & = & \{x \in \Omega,\ p(x) = 0\} & \text{(cavitated region)},\\ \Sigma & = & \partial\Omega^+ \cup \Omega & \text{(free boundary)}.\end{array}$$

Thus, working in dimensionless data ($6\mu_0 v_0 = 1$ for instance), the piezoviscous elastohydrodynamic problem can be written as

$$(\mathcal{P}_\theta)\begin{cases}\text{Find } (p,\theta) \in V \times L^\infty(\Omega) \text{ such that:}\\ \displaystyle\int_\Omega h^3[p]\ e^{-\alpha p}\ \nabla p\ \nabla\phi = \int_\Omega \theta\ h[p]\ \frac{\partial \phi}{\partial x_1} + \int_{\Gamma_\star} \theta_\star\ h[p]\ \phi, \quad \forall\ \phi \in V\\ p \geq 0, \quad p\ (1-\theta) = 0, \quad 0 \leq \theta \leq 1, \quad \text{a.e.},\end{cases}$$

where the functional space V is defined as $V = \big\{v \in H^1(\Omega),\ v_{|\Gamma} = 0\big\}$.

The boundary data, $\theta_\star$, satisfies:

Assumption 9 (Saturation on the boundary)**.**

(i) $\theta_\star \in L^\infty(\Gamma_\star)$,

(ii) $0 \leq \theta_\star(z) \leq 1$, *for a.e.* $z \in \Gamma_\star$.

Finally, let us consider the following technical assumption.

Assumption 10 (Technical hypothesis on the data)**.** *The Sobolev exponent* $r^\star > 2$, *the Sobolev constant* $C(\Omega)$ *(the norm of the trace mapping from* $H^1(\Omega)$ *to* $L^2(\Gamma_\star)$*) and the problem parameters satisfy the condition*

$$\frac{\alpha\ e\ C(\Omega)}{h_0^2}\ \Big|\Omega\Big|^{(1/2)-(1/r^\star)}\ 2^{r^\star/(r^\star-2)} \leq 1, \tag{2.7}$$

We have the following existence theorem:

Theorem 2.3 (Durany, García, Vázquez [DGV96a])**.** *Under Assumptions 6–10, problem* $(\mathcal{P}_\theta)$ *admits at least a solution* (p,θ) *satisfying the following estimates:*

$$\Big\|p\Big\|_{H^1(\Omega)} \leq C_1 \quad \text{and} \quad \Big\|p\Big\|_{L^\infty(\Omega)} \leq C_2,$$

where C_1 *and* C_2 *depend on* α, h_0, h_1, $\widetilde{K}$, $\theta_\star$, Ω, $\Gamma_\star$, $r^\star$.

Remark 2.4. *The complete proof is given by Durany, García and Vázquez [DGV96a]. It is based on the introduction of a penalized problem and Schauder fixed-point theorem. We* *point out the fact that the earlier estimates are not a priori estimates. Thus, we cannot guarantee that each solution of the problem satisfies the earlier estimates. Assumption 10 guarantees an existence result if α is small enough: it holds if the physical configuration is not too far from the isoviscous case.*

Remark 2.5. *Other boundary conditions might be taken into account: a similar result has been proved with Dirichlet conditions on the pressure by Bayada, El Alaoui and Vázquez [BEATV96]. The homogenization study that follows can be easily adapted to this specific type of boundary conditions.*

The next sections deal with homogenization of the lubrication problem, using the periodic unfolding method which has been introduced Cioranescu, Damlamian and Griso [CDG02]. This method has strong links with the two-scale convergence technique which was introduced by Nguetseng [Ngu89], and further developped by Allaire [All92], Lukkassen, Nguetseng and Wall [LNW02].

2.2 Homogenization of the EHL problem

In this section, we state some preliminary results before the homogenization process of the lubrication problem. Let us introduce the microscale domain $Y =]0,1[\times]0,1[$. The nominal gap, i.e. without elastic deformation, is now described by the nominal regular thickness h_r to which one must add the roughness defaults around the average gap. Thus, we consider that the nominal gap is described by:

$$h_r^\varepsilon(x) = h_r(x) + \lambda\left(\frac{x}{\varepsilon}\right),$$

where h_r has been defined in Assumption 6 and λ satisfies:

Assumption 11 (Roughness pattern)**.**

(i) $\lambda \in C_\sharp(Y) = \{v \in C^0(Y),\ v \text{ is } Y \text{ periodic}\}$,

(ii) $\exists\ \lambda_{\max} > 0, \quad \left\|\lambda\right\|_{L^\infty(Y)} \leq \lambda_{\max} < h_0.$

Remark 2.6. *Assumptions 6 and 11 guarantee the uniform coerciveness (with respect to the parameter ε) of the bilinear form; in fact, we have*

$$\forall x \in \Omega, \quad 0 < h_0 - \left\|\lambda\right\|_{L^\infty(Y)} \leq h_r^\varepsilon(x) \leq h_1 + \left\|\lambda\right\|_{L^\infty(Y)}. \tag{2.8}$$

Let us remark that it leads us to consider the roughness of the upper surface, assumed to be fixed, so that the x variable becomes highly oscillating. Thus it means that only the rigid contribution to the gap is rough.

Now let us define the effective gaps:

Definition 2. *For any $q \in L^\infty(\Omega)$, let $h[q]$ and $h_\varepsilon[q]$ be the functions defined by:*

$$\begin{array}{rclcl} h[q] & : & \Omega \times Y & \longrightarrow & \mathbb{R} \\ & & (x,y) & \longrightarrow & h[q](x,y) = h_r(x) + \lambda(y) + \displaystyle\int_\Omega k(x,z)q(z)\ dz, \\ h_\varepsilon[q] & : & \Omega & \longrightarrow & \mathbb{R} \\ & & x & \longrightarrow & h_\varepsilon[q](x) = h[q]\left(x, \dfrac{x}{\varepsilon}\right). \end{array}$$

Thus, we introduce the rough problem:

$$(\mathcal{P}_\theta^\varepsilon) \begin{cases} \text{Find } (p_\varepsilon, \theta_\varepsilon) \in V \times L^\infty(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega h_\varepsilon^3[p_\varepsilon]\ e^{-\alpha p_\varepsilon}\ \nabla p_\varepsilon\ \nabla\phi = \int_\Omega \theta_\varepsilon\ h_\varepsilon[p_\varepsilon]\ \frac{\partial \phi}{\partial x_1} + \int_{\Gamma_\star} \theta_\star\ h_\varepsilon[p_\varepsilon]\ \phi, \quad \forall\ \phi \in V \\ p_\varepsilon \geq 0, \quad p_\varepsilon\ (1-\theta_\varepsilon) = 0, \quad 0 \leq \theta_\varepsilon \leq 1, \quad \text{a.e.} \end{cases}$$

In order to get an existence theorem, we adapt the assumptions to the rough problem. Therefore, Assumption 10 is replaced by:

Assumption 12. *The Sobolev exponent $r^\star > 2$, the Sobolev constant $C(\Omega)$ (the norm of the trace mapping from $H^1(\Omega)$ to $L^2(\Gamma_\star)$) and the problem parameters satisfy the condition*

$$\frac{\alpha\ e\ C(\Omega)}{\left(h_0 - \Big\|\lambda\Big\|_{L^\infty(\Omega)}\right)^2} \Big|\Omega\Big|^{(1/2)-(1/r^\star)}\ 2^{r^\star/(r^\star-2)} \leq 1. \tag{2.9}$$

Thus we get:

Theorem 2.7. *Under Assumptions 6–9, 11 and 12, for any $\varepsilon > 0$, problem $(\mathcal{P}_\theta^\varepsilon)$ admits at least a solution $(p_\varepsilon, \theta_\varepsilon)$ satisfying the following estimates:*

$$\Big\|\nabla p_\varepsilon\Big\|_{L^2(\Omega)} \leq C_3, \quad \Big\|p_\varepsilon\Big\|_{L^\infty(\Omega)} \leq C_4, \quad \Big\|\theta_\varepsilon\Big\|_{L^\infty(\Omega)} \leq 1, \tag{2.10}$$

where C_3 and C_4 only depend on α, $h_0 - \|\lambda\|_{L^\infty(\Omega)}$, h_1, $\widetilde{K}$, $\theta_\star$, Ω, $\Gamma_\star$, $r^\star$.

Remark 2.8. *In mechanical applications (ball or linear bearings), typical roughness is assumed to be either transverse or longitudinal. However, such an assumption on the roughness form is not necessary and more general shapes may be introduced.*

Our purpose is to discuss the behaviour of problem $(\mathcal{P}_\theta^\varepsilon)$ when ε goes to 0, using periodi unfolding methods. From now on, we suppose that Assumptions 6–9, 11 and 12 are satisfied, in particular in Subsections 2.2.2–2.2.5.

2.2.1 Preliminaries to the periodic unfolding method

First we recall some useful definitions and results for the periodic unfolding method [CDG02].

Lemma 2.9. *The separable Banach space $L^2(\Omega; C_\sharp(Y))$ is dense in $L^2(\Omega\times Y)$. Moreover, if $f \in L^2(\Omega; C_\sharp(Y))$, then $x \mapsto \sigma_\varepsilon(f)(x) = f(x, x/\varepsilon)$ is a measurable function such that*

$$\left\|\sigma_\varepsilon(f)\right\|_{L^2(\Omega)} \leq \left\|f\right\|_{L^2(\Omega; C_\sharp(Y))}.$$

Let us now briefly introduce periodic unfolding methods [CDG02]:

Definition 3 (Unfolding operator)**.** *For any $x \in \mathbb{R}^2$, let be $[x]_Y$ the unique integer combination $\sum_{j=1}^n k_j b_j$ of the periods such that $x - [x]_Y$ belongs to Y. We also define $\{x\}_Y = x - [x]_Y \in Y$, so that, for each $x \in \mathbb{R}^2$, one has*

$$x = \varepsilon\, \left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right).$$

Then the unfolding operator $\mathcal{T}_\varepsilon : L^2(\Omega) \to L^2(\Omega\times Y)$ is defined as follows: for $w \in L^2(\Omega)$, extended by zero outside Ω,

$$\mathcal{T}_\varepsilon(w)(x,y) = w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\ y\right), \quad \text{for } x \in \Omega \text{ and } y \in Y.$$

Let us now recall Propositions 1 and 2 of Cioranescu, Damlamian and Griso [CDG02]:

Proposition 2.10. *One has the following integration formula*

$$\int_\Omega w = \int_{\Omega\times Y} \mathcal{T}_\varepsilon(w), \quad \forall\ w \in L^1(\Omega).$$

Proposition 2.11. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. Then the following propositions are equivalent:*

(i) $\mathcal{T}_\varepsilon(u_\varepsilon)$ weakly converges to u_0 in $L^2(\Omega\times Y)$.

(ii) u_ε two-scale converges to u_0.

Similarly to the two-scale convergence technique, the following properties hold:

Lemma 2.12.

(i) Let u_ε be a bounded sequence in $L^2(\Omega)$. Then there exists $u_0 \in L^2(\Omega\times Y)$ such that, up to a subsequence, $\mathcal{T}_\varepsilon(u_\varepsilon)$ weakly converges to u_0 in $L^2(\Omega\times Y)$.

(ii) Let u_ε be a bounded sequence in $H^1(\Omega)$, which weakly converges to a limit $u_0 \in H^1(\Omega)$. Then $\mathcal{T}_\varepsilon(u_\varepsilon)$ weakly converges to u_0 in $L^2(\Omega \times Y)$ and there exists a function $u_1 \in L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$ such that, up to a subsequence, $\mathcal{T}_\varepsilon(\nabla u_\varepsilon)$ weakly converges to $\nabla u_0 + \nabla_y u_1$ in $L^2(\Omega \times Y)$.

To conclude this brief introduction, we have the following (straightforward) property:

Proposition 2.13. *Let $u \in L^2(\Omega)$. If a sequence $u_\varepsilon \in L^2(\Omega)$ strongly converges to u in $L^2(\Omega)$, then $\mathcal{T}_\varepsilon(u_\varepsilon)$ strongly converges to u in $L^2(\Omega \times Y)$.*

2.2.2 Convergence results for problem $(\mathcal{P}^\varepsilon_\theta)$

Lemma 2.14. *There exists $p_0 \in V$ such that, up to a subsequence:*

$$p_\varepsilon \rightharpoonup p_0 \text{ in } H^1(\Omega) \qquad \text{and} \qquad p_\varepsilon \rightarrow p_0 \text{ in } L^2(\Omega).$$

We have also the following convergences:

(i) $\mathcal{T}_\varepsilon(p_\varepsilon)$ strongly converges to p_0 in $L^2(\Omega \times Y)$. Moreover, there exists a function $p_1 \in L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$ and a subsequence ε', still denoted ε, such that $\mathcal{T}_\varepsilon(\nabla p_\varepsilon)$ weakly converges to $\nabla p_0 + \nabla_y p_1$ in $L^2(\Omega \times Y)$.

(ii) There exists $\theta_0 \in L^2(\Omega \times Y)$ and a subsequence ε", still denoted ε, such that $\mathcal{T}_\varepsilon(\theta_\varepsilon)$ weakly converges to θ_0 in $L^2(\Omega \times Y)$.

Moreover, $p_0 \geq 0$ a.e.

Proof. The convergence results are the consequence of Estimates 2.10 (see Theorem 2.7), which do not depend on ε, and Lemma 2.12. □

Remark 2.15. *The following proposition has been stated in the hydrodynamic case (see Chapter 1). The same proof can be used, but we propose an alternate proof, based on periodic unfolding methods. However, the idea remains the same.*

Proposition 2.16. $0 \leq \theta_0 \leq 1$ *and* $p_0\ (1 - \theta_0) = 0$ *a.e.*

Proof.
■ *1st step* - As $0 \leq \theta_\varepsilon$ a.e. and using the definition of the unfolding operator, one has that $0 \leq \mathcal{T}_\varepsilon(\theta_\varepsilon)$ a.e. By Proposition 2.11, one knows that $\mathcal{T}_\varepsilon(\theta_\varepsilon)$ weakly converges to θ_0 in $L^2(\Omega \times Y)$. Thus one has

$$\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\theta_\varepsilon)\ \phi \longrightarrow \int_{\Omega\times Y} \theta_0\ \phi, \quad \forall \phi \in L^2(\Omega \times Y). \tag{2.11}$$

We rewrite θ_0 as $\theta_0 = \theta_0^+ - \theta_0^-$ (with $w^+ = \max(w,0)$ and $w^- = -\min(w,0)$, for any $w \in L^2(\Omega \times Y)$). Then using θ_0^- as a test function in Equation (2.11):

$$A_\varepsilon = \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\theta_\varepsilon)\ \theta_0^- \longrightarrow - \int_{\Omega\times Y} (\theta_0^-)^2 \ = A \leq 0.$$

Since $\mathcal{T}(\theta_\varepsilon) \geq 0$ a.e., A_ε is a sequence of positive numbers converging to a non positive number. Then $A = 0$ and $\theta_0^- = 0$ a.e. The same method is used to prove $0 \leq 1 - \theta_0$ a.e.

■ *2nd step* - The result is also easily obtained using periodic unfolding methods. Indeed by Lemma 2.14 and Proposition 2.11,

$$\begin{array}{rcll} \mathcal{T}_\varepsilon(p_\varepsilon) & \to & p_0, & \text{in } L^2(\Omega\times Y), \\ \mathcal{T}_\varepsilon(\theta_\varepsilon) & \rightharpoonup & \theta_0, & \text{in } L^2(\Omega\times Y). \end{array}$$

Thus one gets:

$$\int_{\Omega\times Y} \mathcal{T}_\varepsilon\left(p_\varepsilon\ (1-\theta_\varepsilon)\right) = \int_{\Omega\times Y} \mathcal{T}_\varepsilon(p_\varepsilon)\ \mathcal{T}_\varepsilon(1-\theta_\varepsilon) \longrightarrow \int_{\Omega\times Y} p_0\ (1-\theta_0).$$

Since $p_\varepsilon(1-\theta_\varepsilon) = 0$, passing to the limit gives

$$\int_{\Omega\times Y} p_0\ (1-\theta_0) = 0.$$

Moreover since $p_0 \geq 0$ a.e. and $1 - \theta_0 \geq 0$ a.e. (see Lemma 2.14 and the first part of Proposition 2.16), the proof is concluded. □

Lemma 2.17. *For $1 \leq p \leq +\infty$,*

(i) $\displaystyle\int_\Omega k(\cdot,z)\, p_\varepsilon(z)\, dz$ *strongly converges to* $\displaystyle\int_\Omega k(\cdot,z)\, p_0(z)\, dz$ *in* $L^p(\Omega)$*,*

(ii) $\displaystyle\int_\Omega k(\cdot,z)\, p_\varepsilon(z)\, dz$ *strongly converges to* $\displaystyle\int_\Omega k(\cdot,z)\, p_0(z)\, dz$ *in* $L^p(\Gamma_\star)$*.*

Proof.
■ *Proof of* (*i*) - Since k satisfies Equation (2.4), one gets by Lemma 1 of Oden and Wu [OW85]:

$$\max_{x\in\overline{\Omega}} \left| \int_\Omega k(x,z)(p_\varepsilon(z) - p_0(z))\ dz \right| \leq C_5(k)\ \left\| p_\varepsilon - p_0 \right\|_{L^q(\Omega)}, \tag{2.12}$$

for any q which can be written as $q = (2-s)/(1-s) > 2$ with $0 < s < 1$. Moreover by Theorem IX.16 (Rellich-Kondrachov) in [Bre83], one has $H^1(\Omega) \hookrightarrow L^r(\Omega)$, $\forall\, r \in [1, +\infty[$.

Since $p_\varepsilon \rightharpoonup p_0$ in $H^1(\Omega)$ (see Lemma 2.14), then $p_\varepsilon \to p_0$ in $L^q(\Omega)$ and

$$\Big\| \int_\Omega k(\cdot, z)(p_\varepsilon(z) - p_0(z))\ dz \Big\|_{L^\infty(\Omega)} \longrightarrow 0.$$

At last, we gain (for $1 \leq p < +\infty$)

$$\Big\| \int_\Omega k(\cdot, z)(p_\varepsilon(z) - p_0(z))\ dz \Big\|_{L^p(\Omega)} \leq \Big|\Omega\Big|^{1/p} \Big\| \int_\Omega k(\cdot, z)(p_\varepsilon(z) - p_0(z))\ dz \Big\|_{L^\infty(\Omega)},$$

the result is proved.

■ *Proof of (ii)* - For $p = +\infty$, the result is immediatly obtained from Inequality (2.12). For $1 \leq p < +\infty$, let us compute $u_\varepsilon = \Big\| \int_\Omega k\big(\cdot, z\big)\ \Big(p_\varepsilon(z) - p_0(z)\Big)\ dz \Big\|_{L^p(\Gamma_\star)}$.

$$\begin{aligned} u_\varepsilon &= \Big(\int_{\Gamma_\star} \Big[\int_\Omega k\,(s, z)\ \big(p_\varepsilon(z) - p_0(z)\big)\ dz \Big]^p\ ds \Big)^{1/p} \\ &\leq \Big|\Gamma_\star\Big|^{1/p}\ \max_{x \in \overline{\Omega}} \Big| \int_\Omega k\big(\cdot, z\big)\ \big(p_\varepsilon(z) - p_0(z)\big)\ dz \Big|, \end{aligned}$$

and using Inequality (2.12) gives

$$u_\varepsilon \leq \Big|\Gamma_\star\Big|^{1/p}\ C_5(k)\ \Big\|p_\varepsilon - p_0\Big\|_{L^q(\Omega)},$$

so that u_ε exists and tends to 0. □

Now we analyse the convergence of each term of problem $(\mathcal{P}_\theta^\varepsilon)$:

Lemma 2.18. *One has the following strong convergences in* $L^2(\Omega \times Y)$*:*

(i) $\mathcal{T}_\varepsilon\left(h_\varepsilon[p_\varepsilon]\right) \longrightarrow h[p_0]$,

(ii) $\mathcal{T}_\varepsilon\left(h_\varepsilon^3[p_\varepsilon]\right) \longrightarrow h^3[p_0]$,

(iii) $\mathcal{T}_\varepsilon\left(e^{-\alpha p_\varepsilon}\right) \longrightarrow e^{-\alpha p_0}$.

Moreover, as a direct consequence of (ii) and (iii), one has:

(iv) $\mathcal{T}_\varepsilon\left(h_\varepsilon^3[p_\varepsilon]\ e^{-\alpha p_\varepsilon}\right) \longrightarrow h^3[p_0]\ e^{-\alpha p_0}$.

Proof.

■ *Proof of (i)* - $\mathcal{T}_\varepsilon\left(h_\varepsilon^3[p_\varepsilon]\right)$ strongly converges to $h^3[p_0]$ in $L^2(\Omega \times Y)$. Indeed, using the definition and linearity of the unfolding operator, we have:

$$\mathcal{T}_\varepsilon\left(h_\varepsilon[p_\varepsilon]\right)(x, y) = h_r(x, y) + \mathcal{T}_\varepsilon \left(\int_\Omega k(x, z)\ p_\varepsilon(z)\ dz \right).$$

Using the property of the Hertz kernel (especially Lemma 2.17 and Proposition 2.13), we easily state that

$$\mathcal{T}_\varepsilon\left(\int_\Omega k(x,z)\ p_\varepsilon(z)\ dz\right) \longrightarrow \int_\Omega k(x,z)\ p_0(z)\ dz, \quad \text{in } L^2(\Omega\times Y).$$

- *Proof of (ii)* - Obviously, $\mathcal{T}_\varepsilon\left(h_\varepsilon[p_\varepsilon]\right)$ and $h[p_0]$ have the following L^∞ bound:

$$C_6 = 3\big(h_1 + \|\lambda\|_{L^\infty(\Omega)} + \widetilde{K}C_4\big)^2,$$

which does not depend on ε. Now, using the definition of the unfolding operator, we have:

$$\int_{\Omega\times Y}\left(\mathcal{T}_\varepsilon\left(h_\varepsilon^3[p_\varepsilon]\right) - h^3[p_0]\right)^2 \quad\leq\quad 9C_6^4\int_{\Omega\times Y}\left(\mathcal{T}_\varepsilon\left(h_\varepsilon[p_\varepsilon]\right) - h[p_0]\right)^2$$

which tends to 0, by the result stated at (i).

- *Proof of (iii)* - By Proposition 2.13, it is sufficient to prove that $e^{-\alpha p_\varepsilon}$ strongly converges to $e^{-\alpha p_0}$ in $L^2(\Omega)$. For this, let us point out the fact that $z\mapsto e^{-\alpha z}$ is a Lipschitz continuous function on $\mathbb{R}^+$, α being a Lipschitz constant. Thus, we have

$$\int_\Omega (e^{-\alpha p_\varepsilon} - e^{-\alpha p_0})^2 \leq \alpha^2\int_\Omega (p_\varepsilon - p_0)^2$$

which tends to 0 and concludes this proof.

- *Proof of (iv)* - By using the two previous results, the convergence result is stated due to the fact that each term is bounded in $L^\infty(\Omega)$.

□

Now, we are interested in the convergence of the boundary term:

Lemma 2.19. *Let γ denote the trace operator and let us define*

$$\widehat{h[p_0]} = \gamma\left(h_r(\cdot) + \int_0^1 \lambda((0,y_2))\ dy_2 + \int_\Omega k(\cdot,z)\ p_0(z)\ dz\right).$$

Then, one has:

$$\gamma\left(h_\varepsilon[p_\varepsilon]\right) \rightharpoonup \widehat{h[p_0]} \ \text{ in } L^2(\Gamma_\star).$$

Proof. In the boundary integral of problem $(\mathcal{P}_\theta^\varepsilon)$, $h_\varepsilon[p_\varepsilon]$ has to be taken in the sense of traces. Thus, since we have

$$h_\varepsilon[p_\varepsilon](x) = h_r(x) + \lambda\left(\frac{x}{\varepsilon}\right) + \int_\Omega k(x,z)\ p_\varepsilon(z)\ dz,$$

it can be written as the sum of a function which belongs to $L^\infty(\Gamma_\star)$, namely

$$\gamma\left(h_r(\cdot) + \int_\Omega k(\cdot, z)\ p_\varepsilon(z)\ dz\right),$$

and the trace of the oscillating function $x \mapsto \sigma_\varepsilon(\lambda)(x) = \lambda(x/\varepsilon)$ (according to the definition of the operator σ_ε given in Lemma 2.9), i.e.

$$\gamma(\sigma_\varepsilon(\lambda)(\cdot)).$$

Thus, let us study the convergence of each term with respect to ε:

- First, the following convergence holds:

$$\gamma\left(h_r(\cdot) + \int_\Omega k(\cdot, z)\ p_\varepsilon(z)\ dz\right) \longrightarrow \gamma\left(h_r(\cdot) + \int_\Omega k(\cdot, z)\ p_0(z)\ dz\right), \quad \text{in } L^2(\Gamma_\star).$$

  Indeed, by linearity, the difference of these two terms is equal to the trace of

$$\int_\Omega k(\cdot, z)\ (p_\varepsilon(z) - p_0(z))\ dz$$

  which strongly converges to 0 in $L^2(\Gamma_\star)$ by Lemma 2.17.

- Next, using the assumptions on the roughness regularity, $\gamma(\sigma_\varepsilon(\lambda))$ can be identified to the function $x_2 \mapsto \lambda((0, x_2/\varepsilon))$, which weakly converges in $L^2(]0,1[)$ to its average with respect to y_2, namely the constant

$$\int_0^1 \lambda((0, y_2))\ dy_2.$$

Thus the proof is achieved. □

2.2.3 Homogenization of the EHL problem (general case)

Once we have obtained the limits of the different terms which appear in problem $(\mathcal{P}_\theta^\varepsilon)$, we state, as usual with the two-scale convergence technique, the macroscopic and microscopic equations for the homogenized problem.

Proposition 2.20. *The limit functions p_0, p_1 and θ_0 satisfy the following equations:*

- *Macroscopic equation - For every ϕ in V,*

$$\int_\Omega \left(\int_Y h^3[p_0]\ e^{-\alpha p_0}\ \Big[\nabla p_0 + \nabla_y p_1\Big]\right)\ \nabla\phi = \int_\Omega \left(\int_Y \theta_0\ h[p_0]\right) \frac{\partial \phi}{\partial x_1} + \int_{\Gamma_\star} \theta_\star\ \widehat{h[p_0]}\ \phi.$$

■ *Microscopic equation - For a.e. $x \in \Omega$, for every $\psi \in H^1_\sharp(Y)$,*

$$\int_Y h^3[p_0](x,\cdot)\ e^{-\alpha p_0(x)} \Big[\nabla p_0(x) + \nabla_y p_1(x,\cdot)\Big]\ \nabla_y \psi = \int_Y\ \theta_0(x,\cdot)\ h[p_0](x,\cdot)\ \frac{\partial \psi}{\partial y_1}.$$

Proof. Considering problem $(\mathcal{P}^\varepsilon_\theta)$, we have, using Proposition 2.10:

$$\int_{\Omega\times Y} \mathcal{T}_\varepsilon \left(h^3_\varepsilon[p_\varepsilon]\ e^{-\alpha p_\varepsilon}\right)\ \mathcal{T}_\varepsilon\left(\nabla p_\varepsilon\right)\ \mathcal{T}_\varepsilon\left(\nabla \varphi_\varepsilon\right)$$
$$= \int_{\Omega\times Y}\ \mathcal{T}_\varepsilon(\theta_\varepsilon)\ \mathcal{T}_\varepsilon(h_\varepsilon[p_\varepsilon])\ \mathcal{T}_\varepsilon\left(\frac{\partial \varphi_\varepsilon}{\partial x_1}\right) + \int_{\Gamma^\star} \theta_\star(r)\ h_\varepsilon[p_\varepsilon](r)\ \varphi_\varepsilon(r)\ dr.$$

for all $\varphi_\varepsilon \in V$. Taking a test function $x \mapsto \phi(x)$, with $\phi \in V$ and using the convergence results stated in Lemmas 2.14, 2.18 and 2.19 gives the macroscopic equation by passing to the limit with respect to ε. Now taking the test function $x \mapsto \varepsilon\ \phi(x)\ \psi(x/\varepsilon)$, with $\phi \in \mathcal{D}(\Omega)$ and $\psi \in H^1_\sharp(Y)$, gives the microscopic one. □

Definition 4. *For a given $p_0 \in L^\infty(\Omega)$, let us define the following functions:*

$$\begin{aligned} a[p_0](x,y) &= h^3[p_0](x,y), \quad (x,y) \in \Omega \times Y, \\ b[p_0](x,y) &= h[p_0](x,y), \quad (x,y) \in \Omega \times Y. \end{aligned}$$

Let us define the local problems, respectively denoted $(\mathcal{M}^\star_i)$, $(\mathcal{N}^\star_i)$ and $(\mathcal{N}^0_i)$:

Find $W^\star_i$, $\chi^\star_i$, χ^0_i $(i = 1,2)$ in $L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$, such that, for a.e. $x \in \Omega$:

$$\int_Y a[p_0](x,\cdot)\ \nabla_y W^\star_i(x,\cdot)\ \nabla_y \psi = \int_Y a[p_0](x,\cdot)\ \frac{\partial \psi}{\partial y_i}, \tag{2.13}$$

$$\int_Y a[p_0](x,\cdot)\ \nabla_y \chi^\star_i(x,\cdot)\ \nabla_y \psi = \int_Y b[p_0](x,\cdot)\ \frac{\partial \psi}{\partial y_i}, \tag{2.14}$$

$$\int_Y a[p_0](x,\cdot)\ \nabla_y \chi^0_i(x,\cdot)\ \nabla_y \psi = \int_Y \theta_0(x,\cdot)\ b[p_0](x,\cdot)\ \frac{\partial \psi}{\partial y_i}, \tag{2.15}$$

for all $\psi \in H^1_\sharp(Y)$. We immediatly have the following proposition:

Proposition 2.21. *The local problem $(\mathcal{M}^\star_i)$ (resp. $(\mathcal{N}^\star_i)$,$(\mathcal{N}^0_i)$) admits a unique solution $W^\star_i$ (resp. $\chi^\star_i$, χ^0_i) in $L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$.*

Theorem 2.22. *The homogenized problem can be written as:*

$$(\mathcal{P}^\star_\theta) \left\{ \begin{array}{l} \textit{Find } (p_0, \Theta_1, \Theta_2) \in V \times L^\infty(\Omega) \times L^\infty(\Omega) \textit{ such that:} \\ \displaystyle\int_\Omega e^{-\alpha p_0}\ \mathcal{A}[p_0] \cdot \nabla p_0\ \nabla \phi = \int_\Omega \underline{b}^0[p_0]\ \nabla \phi + \int_{\Gamma_\star} \theta_\star\ \widehat{h[p_0]}\ \phi, \quad \forall\ \phi \in V \\ p_0 \geq 0, \quad p_0\ (1 - \Theta_i) = 0,\ (i = 1,2) \quad a.e., \end{array} \right.$$

with $\widetilde{f}(x) = \displaystyle\int_Y f(x,y)\ dy$, *and*

$$\mathcal{A}[p_0] \;=\; \begin{pmatrix} \widetilde{a[p_0]} - \left[a[p_0]\widetilde{\dfrac{\partial W_1^\star}{\partial y_1}}\right] & -\left[a[p_0]\widetilde{\dfrac{\partial W_2^\star}{\partial y_1}}\right] \\ -\left[a[p_0]\widetilde{\dfrac{\partial W_1^\star}{\partial y_2}}\right] & \widetilde{a[p_0]} - \left[a[p_0]\widetilde{\dfrac{\partial W_2^\star}{\partial y_2}}\right] \end{pmatrix},$$

$$\underline{b}^0[p_0] \;=\; \begin{pmatrix} \Theta_1[p_0]\ b_1^\star[p_0] \\ \Theta_2[p_0]\ b_2^\star[p_0] \end{pmatrix},$$

with the notations ($i = 1,2$)*:*

$$b_i^\star[p_0] = \widetilde{b[p_0]} - \left[a[p_0]\widetilde{\frac{\partial \chi_i^\star}{\partial y_i}}\right], \qquad b_i^0[p_0] = \left[\widetilde{\theta_0 b[p_0]}\right] - \left[a[p_0]\widetilde{\frac{\partial \chi_i^0}{\partial y_i}}\right],$$

and defining the following ratios ($i = 1,2$)*:*

$$\Theta_i[p_0] = \frac{b_i^0[p_0]}{b_i^\star[p_0]}.$$

Proof. From the local problems, we easily obtain in $L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$:

$$p_1(x,y) = -\begin{pmatrix} W_1^\star(x,y) \\ W_2^\star(x,y) \end{pmatrix} \cdot \nabla p_0(x) + e^{\alpha p_0(x)}\ \chi_1^0(x,y).$$

The homogenized problem follows by replacing the previous expression of p_1 in the macroscopic equation. □

Remark 2.23. *The homogenized lubrication problem can be considered as a generalized elastohydrodynamic Reynolds-type problem with two cavitation parameters* Θ_i ($i = 1,2$)*. Let us notice the fact that we do not have the property* $0 \leq \Theta_i \leq 1$*, i.e. we cannot guarantee that homogenized cavitation parameters are smaller than 1 in cavitation areas ! Thus, at that point, the homogenized problem does not have a structure similar to the initial one. But, in the next subsections, we prove the following additional results:*

- *in Subsection 2.2.4, we state that, among the solutions of the homogenized problem, there exists a class of solutions with isotropic saturation, that is, the homogenized problem* $(\mathcal{P}_\theta^\star)$ *admits a solution* (p_0, Θ, Θ) *with* $p_0 \geq 0$ *and* $p_0\ (1-\Theta) = 0$ *and also the additive property (which lacks in the formulation of the homogenized problem* $(\mathcal{P}_\theta^\star)$ *in the general case):* $0 \leq \Theta \leq 1$ *a.e.*

- *in Subsection 2.2.5, we state that, under additional assumptions on the roughness pattern, only one single saturation function* Θ *appears in the homogenized problem. Moreover, it satisfies* $0 \leq \Theta \leq 1$ *a.e in* Ω*.*

2.2.4 Existence of solutions with isotropic saturation

This subsection is devoted to the proof of the following theorem:

Theorem 2.24. *The homogenized problem* $(\mathcal{P}^\star_\theta)$ *admits a solution* (p_0, Θ, Θ) *with the property* $0 \leq \Theta \leq 1$ *(and also* $p_0 \geq 0$*,* $p_0\ (1-\Theta) = 0$*) a.e.*

Theorem 2.24 guarantees the existence of solutions with isotropic saturation Θ. Moreover, the saturation satisfies the property $0 \leq \Theta \leq 1$, which lacks in the general formulation of the homogenized problem. The result is obtained in the following three steps which are based on the existence result and the corresponding method used in Durany, García, Vázquez [DGV96a]:

- *1st step: Introduction of a penalized problem,*
- *2nd step: Homogenization of the penalized problem,*
- *3rd step: Convergence with respect to the penalized parameter.*

Remark 2.25. *Interestingly, in the earlier scheme, forgetting the 2nd step (i.e. omitting the homogenization step) would lead us to the existence result for problem* $(\mathcal{P}^\varepsilon_\theta)$*, namely Theorem 2.7. Thus, the reader should not be surprised to see that constants which have been already used or defined in Theorem 2.7 appear in the details of the forthcoming proof.*

For convenience, these three steps are given in details and the idea of the proof is sketched at the end of this subsection. It can be noticed that the general frame is quite similar than in the hydrodynamic case developped in Chapter 1. Nevertheless, it is much more difficult from a technical point of view, as it will be pointed out. In particular, the 3rd step needs further analysis due to the nonlocal term and the piezoviscous one.

- *1st step: Introduction of a penalized problem*

As in the smooth case studied by Durany, García, Vázquez [DGV96a], we introduce the following ε dependent penalized problem:

$$(\mathcal{P}^\varepsilon_\eta) \begin{cases} \textit{Find } p^\eta_\varepsilon \in V \textit{ such that:} \\ \displaystyle\int_\Omega h^3_\varepsilon[p^\eta_\varepsilon]\ e^{-\alpha p^\eta_\varepsilon}\ \nabla p^\eta_\varepsilon\ \nabla\phi = \int_\Omega H_\eta(p^\eta_\varepsilon)\ h_\varepsilon[p^\eta_\varepsilon]\ \frac{\partial\phi}{\partial x_1} + \int_{\Gamma_\star} \theta_\star\ h_\varepsilon[p^\eta_\varepsilon]\ \phi, \quad \forall\ \phi \in V \\ p^\eta_\varepsilon \geq 0, \quad \text{a.e.,} \end{cases}$$

where the function H_η is the usual approximation of the Heaviside graph (see Chapter 1). The application of Theorem 3.2. of Durany, García, Vázquez [DGV96a], which is based on a fixed point technique leads to the following results:

Theorem 2.26. *For every $\eta > 0$, problem $(\mathcal{P}^\varepsilon_\eta)$ admits a positive solution. Moreover, we can obtain the following (ε, η) independent estimates:*

$$\left\| p^\eta_\varepsilon \right\|_{H^1(\Omega)} \leq C_3, \quad \left\| p^\eta_\varepsilon \right\|_{L^\infty(\Omega)} \leq C_4. \tag{2.16}$$

Remark 2.27. *We point out the fact that Theorem 2.26 holds under Assumptions 6–9, 11 and 12 which are implicitely imposed as in previous subsections. In particular, Assumption 12 is necessary to allow the use of a fixed point technique and obtain Theorem 2.26. In particular, the L^∞ estimates play an important role in the proof of existence of isotropic solutions.*

■ *2nd step: Homogenization of the penalized problem*

We proceed to the homogenization of problem $(\mathcal{P}^\varepsilon_\eta)$ with respect to ε: from Estimates (2.16) and using the periodic unfolding method (as in the previous subsection), we immediatly get the following convergence results and macro/microscopic decomposition (for convenience, proofs are omitted when they are close to the ones stated in previous subsections):

Proposition 2.28. *There exists $p^\eta_0 \in V$ ($p^\eta_0 \geq 0$ a.e.), $p_1 \in L^2(\Omega; H_\sharp(Y)/\mathbb{R})$ such that, up to a subsequence,*

(i) p^η_ε weakly converges to p^η_0 in $H^1(\Omega)$,

(ii) $\mathcal{T}_\varepsilon(\nabla p^\eta_\varepsilon)$ weakly converges to $\nabla p^\eta_0 + \nabla_y p^\eta_1$ in $L^2(\Omega \times Y)$.

Moreover, we have:

- *Macroscopic equation - For every ϕ in V,*

$$\int_\Omega \Big(\int_Y h^3[p^\eta_0] \, e^{-\alpha p^\eta_0} \left[\nabla p^\eta_0 + \nabla_y p^\eta_1 \right] \Big) \nabla \phi = \int_\Omega \Big(\int_Y H_\eta(p^\eta_0) \, h[p^\eta_0] \Big) \frac{\partial \phi}{\partial x_1} + \int_{\Gamma_\star} \theta_\star \widehat{h[p^\eta_0]} \, \phi.$$

- *Microscopic equation - For a.e. $x \in \Omega$, for every $\psi \in H^1_\sharp(Y)$,*

$$\int_Y h^3[p_0^\eta](x,\cdot)\ e^{-\alpha p_0^\eta(x)} \left[\nabla p_0^\eta(x) + \nabla_y p_1^\eta(x,\cdot)\right] \nabla_y \psi = \int_Y H_\eta(p_0^\eta(x))\ h[p_0^\eta](x,\cdot)\ \frac{\partial \psi}{\partial y_1}.$$

Then, recalling the definition of $a[\cdot]$ and $b[\cdot]$ (see Definition 4), and introducing the local problems, respectively denoted $(\mathcal{M}_i^\eta)$, $(\mathcal{N}_i^\eta)$:

Find W_i^η, χ_i^η in $L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$, such that, for a.e. $x \in \Omega$:

$$\int_Y a[p_0^\eta](x,\cdot)\ \nabla_y W_i^\eta(x,\cdot)\ \nabla_y \psi = \int_Y a[p_0^\eta](x,\cdot)\ \frac{\partial \psi}{\partial y_i}, \tag{2.17}$$

$$\int_Y a[p_0^\eta](x,\cdot)\ \nabla_y \chi_i^\eta(x,\cdot)\ \nabla_y \psi = \int_Y b[p_0^\eta](x,\cdot)\ \frac{\partial \psi}{\partial y_i}, \tag{2.18}$$

for all $\psi \in H^1_\sharp(Y)$. We can state:

Lemma 2.29. *The homogenized penalized problem is*

$$(\mathcal{P}_\eta^\star) \begin{cases} \textit{Find } p_0^\eta \in V \textit{ such that:} \\ \displaystyle\int_\Omega e^{-\alpha p_0^\eta}\ \mathcal{A}^\eta[p_0^\eta] \cdot \nabla p_0^\eta\ \nabla \phi = \int_\Omega H_\eta(p_0^\eta)\ \underline{b}^\eta[p_0^\eta]\ \nabla \phi + \int_{\Gamma_\star} \theta_\star\ \widehat{h[p_0^\eta]}\ \phi, \quad \forall\ \phi \in V \\ p_0^\eta \geq 0, \quad \text{a.e.,} \end{cases}$$

with $\mathcal{A}^\eta[p_0^\eta] = \widetilde{a[p_0^\eta]}\ I - \widetilde{a[p_0^\eta]\nabla W^\eta}$ *and* $\underline{b}^\eta[p_0^\eta] = \begin{pmatrix} \widetilde{b[p_0^\eta]} - \left(\widetilde{a[p_0^\eta]\dfrac{\partial \chi_1^\eta}{\partial y_1}}\right) \\ -\left(\widetilde{a[p_0^\eta]\dfrac{\partial \chi_1^\eta}{\partial y_2}}\right) \end{pmatrix}$

Proof. The following equality in $L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$ is classically obtained using the local problems:

$$p_1^\eta(x,y) = -W^\eta(x,y) \cdot \nabla p_0^\eta(x) + e^{\alpha p_0^\eta(x)}\ H_\eta(p_0^\eta(x))\ \chi_1^\eta(x,y). \tag{2.19}$$

Using Equation (2.19) in the macroscopic equation gives us:

$$\begin{aligned} &\int_\Omega e^{-\alpha p_0^\eta} \left[\widetilde{a[p_0^\eta]}\ I - \widetilde{a[p_0^\eta]\nabla W^\eta}\right] \cdot \nabla p_0^\eta\ \nabla \phi \\ &= \int_\Omega H_\eta(p_0^\eta)\left[\widetilde{b[p_0^\eta]} - \left(\widetilde{a[p_0^\eta]\frac{\partial \chi_1^\eta}{\partial y_1}}\right)\right] \frac{\partial \phi}{\partial x_1} + \int_\Omega H_\eta(p_0^\eta)\left[- \left(\widetilde{a[p_0^\eta]\frac{\partial \chi_1^\eta}{\partial y_2}}\right)\right] \frac{\partial \phi}{\partial x_2} \\ &+ \int_{\Gamma_\star} \theta_\star\ \widehat{h[p_0^\eta]}\ \phi, \end{aligned}$$

for every $\phi \in V$. Then, the proof is concluded. $\square$

■ *3rd step: Behaviour of the homogenized penalized problem with respect to η*

Now we study the behaviour of the homogenized penalized problem when η tends to 0.

Proposition 2.30. *There exists $p_0 \in V$ and $\Theta \in L^\infty(\Omega)$ such that*

$$p_0^\eta \rightharpoonup p_0 \text{ in } H^1(\Omega), \qquad H_\eta(p_0^\eta) \rightharpoonup \Theta \text{ in } L^\infty(\Omega) \text{ weak-}\star.$$

Moreover, $p_0 \geq 0$, $0 \leq \Theta \leq 1$ and $p_0\ (1-\Theta) = 0$ a.e.

Proof. The convergences only come from estimates satisfied by p_0^η (see Estimates (2.16)):

$$\left\|p_0^\eta\right\|_{H^1(\Omega)} \leq C_3, \quad \left\|p_0^\eta\right\|_{L^\infty(\Omega)} \leq C_4.$$

The properties and relationships between p_0 and Θ are obtained as in Chapter 1. □

Now we state:

Proposition 2.31. *$e^{-\alpha p_0^\eta}\mathcal{A}^\eta[p_0^\eta]$ strongly converges to $e^{-\alpha p_0}\mathcal{A}[p_0]$ in $L^2(\Omega)$.*

Proof. We prove the result in three steps:

(a) $\|\mathcal{A}^\eta[p_0^\eta]\|_{L^\infty(\Omega)} \leq \widetilde{C}$, $\|\mathcal{A}[p_0]\|_{L^\infty(\Omega)} \leq \widetilde{C}$, where the constant $\widetilde{C}$ does not depend on η,

(b) $\mathcal{A}^\eta[p_0^\eta] \longrightarrow \mathcal{A}[p_0]$, a.e.,

(c) $e^{-\alpha p_0^\eta}\mathcal{A}^\eta[p_0^\eta]$ strongly converges to $e^{-\alpha p_0}\mathcal{A}[p_0]$ in $L^2(\Omega)$.

► Proof of (a):
Let us recall that $\mathcal{A}^\eta[p_0^\eta] = \widetilde{a[p_0^\eta]}I - \widetilde{a[p_0^\eta]\nabla_y W^\eta}$. Obviously, we have $\widetilde{a[p_0^\eta]} \leq C_7$ with

$$C_7 = \left(h_1 + \|\lambda\|_{L^\infty(\Omega)} + \widetilde{K}C_4\right)^3.$$

Thus, we just have to state the estimates for terms of the form

$$\widetilde{a[p_0^\eta]\frac{\partial W_i^\eta}{\partial y_k}}, \quad (i,k = 1,2).$$

Using W_i^η as a test function in the variational formulation of problem $(\mathcal{M}_i^\eta)$ (see Equation (2.17)) gives for a.e. $x \in \Omega$

$$\int_Y a[p_0^\eta](x,\cdot)\left|\nabla_y W_i^\eta(x,\cdot)\right|^2 = \int_Y a[p_0^\eta](x,\cdot)\frac{\partial W_i^\eta}{\partial y_i}(x,\cdot). \tag{2.20}$$

Then, in the left-hand side, we write $a[p_0^\eta]$ as $a^2[p_0^\eta]/a[p_0^\eta]$ and use a lower bound of $1/a[p_0^\eta]$, that is

$$\frac{1}{C_7}\int_Y \left|a[p_0^\eta](x,\cdot)\ \nabla_y W_i^\eta(x,\cdot)\right|^2 \leq \int_Y a[p_0^\eta](x,\cdot)\ \left|\nabla_y W_i^\eta(x,\cdot)\right|^2$$

and this, together with Inequality (2.20), gives

$$\int_Y \left|a[p_0^\eta](x,\cdot)\ \nabla_y W_i^\eta(x,\cdot)\right|^2 \leq |Y|\ C_7^2,$$

which means

$$\left\|a[p_0^\eta](x,\cdot)\ \frac{\partial W_i^\eta}{\partial y_k}(x,\cdot)\right\|_{L^2(Y)}^2 \leq |Y|\ C_7^2, \quad (i,k=1,2).$$

Then, using the Cauchy-Schwarz inequality,

$$\Big|\int_Y a[p_0^\eta](x,\cdot)\ \frac{\partial W_i^\eta}{\partial y_k}(x,\cdot)\Big| \leq |Y|^{1/2}\left\|a[p_0^\eta](x,\cdot)\ \frac{\partial W_i^\eta}{\partial y_k}(x,\cdot)\right\|_{L^2(Y)}$$

that is,

$$\left|\widetilde{a[p_0^\eta]\ \frac{\partial W_i^\eta}{\partial y_k}}(x)\right| \leq |Y|\ C_7, \qquad (i,k=1,2). \tag{2.21}$$

Let us remark that Inequality (2.21) holds for a.e. $x \in \Omega$ so that the matrix

$$\widetilde{a[p_0^\eta]\nabla_y W^\eta}$$

is bounded in $L^\infty(\Omega)$ by a constant which does not depend on η. Thus

$$\mathcal{A}^\eta[p_0^\eta] = \widetilde{a[p_0^\eta]}I - \widetilde{a[p_0^\eta]\nabla_y W^\eta}$$

is bounded by a constant $\widetilde{C}$ which does not depend on η. With the same method, as p_0 satisfies $\|p_0\|_{H^1(\Omega)} \leq C_3$, $\|p_0\|_{L^\infty(\Omega)} \leq C_4$, one proves that

$$\mathcal{A}[p_0] = \widetilde{a[p_0]}I - \widetilde{a[p_0]\nabla_y W^\star}$$

is also bounded by $\widetilde{C}$.

► Proof of (b):

Let us recall that, with the same arguments that have been used in the proof of Lemma 2.17 (see Inequality (2.12)), one has:

$$\int_\Omega k(\cdot,z)\ p_0^\eta(z)\ dz \longrightarrow \int_\Omega k(\cdot,z)\ p_0(z)\ dz \ \text{ in } L^\infty(\Omega).$$

In particular, the convergence holds almost everywhere. Let be x_0 such that:

$$\int_\Omega k(x_0,z)\ p_0^\eta(z)\ dz \longrightarrow \int_\Omega k(x_0,z)\ p_0(z)\ dz.$$

Recalling local problem $(\mathcal{M}_i^\eta)$, one has,

$$\int_Y a[p_0^\eta](x_0,\cdot)\ \nabla_y W_i^\eta(x_0,\cdot)\ \nabla_y\psi = \int_Y a[p_0^\eta](x_0,\cdot)\ \frac{\partial\psi}{\partial y_i}, \quad \forall\psi\in H^1_\sharp(Y).$$

Using $W_i^\eta(x_0,\cdot)$ as a test function and using upper and lower bounds of $a[p_0^\eta]$ gives:

$$\left\|\nabla W_i^\eta(x_0,\cdot)\right\|_{L^2(Y)} \le C_7/C_8$$

with $C_8 = \left(h_0 - \|\lambda\|_{L^\infty(\Omega)}\right)^3$. Thus, $W_i^\eta(x_0,\cdot)$ is bounded in $H^1_\sharp(Y)/\mathbb{R}$ by a constant which does not depend on η. Then, there exists $F_i(x_0,\cdot)\in H^1_\sharp(Y)/\mathbb{R}$ such that $\nabla_y W_i^\eta(x_0,\cdot)$ weakly converges to $\nabla_y F_i(x_0,\cdot)$ in $L^2(Y)$. Moreover, since $a[p_0^\eta](x_0,\cdot)$ strongly converges to $a[p_0](x_0,\cdot)$ in $L^2(Y)$, one has for every $\psi\in H^1_\sharp(Y)$:

$$\begin{array}{lll}\displaystyle\int_Y a[p_0^\eta](x_0,\cdot)\ \nabla_y W_i^\eta(x_0,\cdot)\ \nabla_y\psi & \longrightarrow & \displaystyle\int_Y a[p_0](x_0,\cdot)\ \nabla_y F_i(x_0,\cdot)\ \nabla_y\psi,\\ \displaystyle\int_Y a[p_0^\eta](x_0,\cdot)\ \frac{\partial\psi}{\partial y_i} & \longrightarrow & \displaystyle\int_Y a[p_0](x_0,\cdot)\ \frac{\partial\psi}{\partial y_i}.\end{array}$$

Thus, we get for every $\psi\in H^1_\sharp(Y)$:

$$\int_Y a[p_0](x_0,\cdot)\ \nabla_y F_i(x_0,\cdot)\ \nabla_y\psi = \int_Y a[p_0](x_0,\cdot)\ \frac{\partial\psi}{\partial y_i}$$

and, by uniqueness of the solution to the local problem,

$$\begin{array}{rcll} W_i^\star(x_0,\cdot) & = & F_i(x_0,\cdot), & \text{in } H^1_\sharp(Y)/\mathbb{R},\\ \nabla_y W_i^\star(x_0,\cdot) & = & \nabla_y F_i(x_0,\cdot), & \text{in } L^2(Y).\end{array}$$

Now, since we have:

$$\begin{array}{rcll} a[p_0^\eta](x_0,\cdot) & \to & a[p_0](x_0,\cdot), & \text{in } L^2(Y), \\ \nabla_y W_i^\eta(x_0,\cdot) & \rightharpoonup & \nabla_y W_i^\star(x_0,\cdot), & \text{in } L^2(Y), \end{array}$$

we easily deduce that:

$$\int_Y a[p_0^\eta](x_0,y)\ \nabla_y W_i^\eta(x_0,y)\ dy \longrightarrow \int_Y a[p_0](x_0,y)\ \nabla_y W_i^\star(x_0,y)\ dy,$$

that is,

$$\left[a[p_0^\eta]\widetilde{\nabla_y W_i^\eta}\right](x_0) \longrightarrow \left[a[p_0]\widetilde{\nabla_y W_i^\star}\right](x_0).$$

Since it is clear that $\widetilde{a[p_0^\eta]}(x_0)$ converges to $\widetilde{a[p_0]}(x_0)$, one has:

$$\widetilde{a[p_0^\eta]}(x_0) - \left[a[p_0^\eta]\widetilde{\nabla_y W_i^\eta}\right](x_0) \longrightarrow \widetilde{a[p_0]}(x_0) - \left[a[p_0]\widetilde{\nabla_y W_i^\star}\right](x_0),$$

which states the result (b).

► Proof of (c):
Let us denote $f^\eta = \mathcal{A}^\eta[p_0^\eta]$, $f = \mathcal{A}[p_0]$ and $g^\eta = (f^\eta - f)^2$. It is clear that:

- $\|g^\eta\|_{L^\infty(\Omega)} \leq 4\widetilde{C}^2$ (see property (a)),
- $g^\eta \longrightarrow 0$ a.e. (see property (b)).

Thus, by the Lebesgue theorem, $\mathcal{A}^\eta[p_0^\eta]$ strongly converges to $\mathcal{A}[p_0]$ in $L^2(\Omega)$. Now let us denote

$$r_\eta = \int_\Omega \left(e^{-\alpha p_0^\eta}\mathcal{A}^\eta[p_0^\eta] - e^{-\alpha p_0}\mathcal{A}[p_0]\right)^2.$$

Then, we have:

$$\begin{array}{rcl} r_\eta & = & \displaystyle\int_\Omega \left(\left(e^{-\alpha p_0^\eta} - e^{-\alpha p_0}\right)\mathcal{A}^\eta[p_0^\eta] + e^{-\alpha p_0}\left(\mathcal{A}^\eta[p_0^\eta] - \mathcal{A}[p_0]\right)\right)^2 \\ & \leq & 2\displaystyle\int_\Omega \left(e^{-\alpha p_0^\eta} - e^{-\alpha p_0}\right)^2 \left(\mathcal{A}^\eta[p_0^\eta]\right)^2 + 2\int_\Omega \left(e^{-\alpha p_0}\right)^2\left(\mathcal{A}^\eta[p_0^\eta] - \mathcal{A}[p_0]\right)^2. \end{array}$$

Since $\|\mathcal{A}^\eta[p_0^\eta]\|_{L^\infty(\Omega)} \leq \widetilde{C}$ (see property (a)) and $\|e^{-\alpha p_0}\|_{L^\infty(\Omega)} \leq 1$, one has:

$$r_\eta \ \leq\ 2\ \widetilde{C}^2\int_\Omega \left(e^{-\alpha p_0^\eta} - e^{-\alpha p_0}\right)^2 + 2\int_\Omega \left(\mathcal{A}^\eta[p_0^\eta] - \mathcal{A}[p_0]\right)^2.$$

Since $x \mapsto e^{-\alpha x}$ is α-Lipschitz continuous on $\mathbb{R}^+$, one has

$$r_\eta \quad \leq \quad 2\ \widetilde{C}^2\ \alpha^2 \int_\Omega \left(p_0^\eta - p_0\right)^2 + 2 \int_\Omega \left(\mathcal{A}^\eta[p_0^\eta] - \mathcal{A}[p_0]\right)^2.$$

Then, as p_0^η (resp. $\mathcal{A}^\eta[p_0^\eta]$) strongly converges to p_0 (resp. $\mathcal{A}[p_0]$) in $L^2(\Omega)$, r_η tends to 0, which concludes the proof.

□

Proposition 2.32. *$\underline{b}^\eta[p_0^\eta]$ strongly converges to $\underline{b}^\star[p_0]$ in $L^2(\Omega)$.*

Proof. The proof is similar to the one of Proposition 2.31. More precisely, we state

(a) $\|\underline{b}^\eta[p_0^\eta]\|_{L^\infty(\Omega)} \leq \widetilde{D}$, $\|\underline{b}^\star[p_0]\|_{L^\infty(\Omega)} \leq \widetilde{D}$, where $\widetilde{D}$ does not depend on η,

(b) $\underline{b}^\eta[p_0^\eta] \longrightarrow \underline{b}^\star[p_0]$ a.e.,

and the proof is concluded with the Lebesgue theorem. □

Proposition 2.33. *$\widehat{h[p_0^\eta]}$ strongly converges to $\widehat{h[p_0]}$ in $L^2(\Gamma_\star)$.*

Proof. The proof is straightforward from the properties of the Hertz kernel (see, in particular, Lemma 2.17). □

Now we conclude this subsection by the proof of Theorem 2.24:

Proof of Theorem 2.24. By Lemma 2.29, there exists $p_0^\eta \in V$ such that

$$\int_\Omega e^{-\alpha p_0^\eta}\ \mathcal{A}^\eta[p_0^\eta] \cdot \nabla p_0^\eta\ \nabla \phi = \int_\Omega H_\eta(p_0^\eta)\ \underline{b}^\eta[p_0^\eta]\ \nabla \phi + \int_{\Gamma_\star} \theta_\star\ \widehat{h[p_0^\eta]}\ \phi, \quad \forall\ \phi \in V.$$

Next, from Propositions 2.30 and 2.31, we have:

$$\begin{array}{ccll} \nabla p_0^\eta & \rightharpoonup & \nabla p_0, & \text{in } L^2(\Omega), \\ e^{-\alpha p_0^\eta} \mathcal{A}^\eta[p_0^\eta] & \rightarrow & e^{-\alpha p_0} \mathcal{A}[p_0], & \text{in } L^2(\Omega), \end{array}$$

so that

$$\int_\Omega e^{-\alpha p_0^\eta}\ \mathcal{A}^\eta[p_0^\eta] \cdot \nabla p_0^\eta\ \nabla \phi \longrightarrow \int_\Omega e^{-\alpha p_0}\ \mathcal{A}[p_0] \cdot \nabla p_0\ \nabla \phi, \quad \forall\ \phi \in V.$$

Moreover, from Propositions 2.30 and 2.32, we have:

$$\begin{array}{ccll} H_\eta(p_0^\eta) & \rightharpoonup & \Theta, & \text{in } L^\infty(\Omega) \text{ weak-}\star, \\ \underline{b}^\eta[p_0^\eta] & \rightarrow & \underline{b}^\star[p_0], & \text{in } L^2(\Omega), \end{array}$$

so that

$$\int_\Omega H_\eta(p_0^\eta)\ \underline{b}^\eta[p_0^\eta]\ \nabla\phi \longrightarrow \int_\Omega \Theta\ \underline{b}^\star[p_0]\ \nabla\phi, \quad \forall\ \phi\in V.$$

Next, by Proposition 2.33, we obtain:

$$\int_{\Gamma_\star} \theta_\star\ \widehat{h[p_0^\eta]}\ \phi \longrightarrow \int_{\Gamma_\star} \theta_\star\ \widehat{h[p_0]}\ \phi, \quad \forall\ \phi\in V.$$

Thus, passing to the limit in the homogenized penalized problem, we get

$$\int_\Omega e^{-\alpha p_0}\ \mathcal{A}[p_0]\cdot\nabla p_0\ \nabla\phi = \int_\Omega \Theta\ \underline{b}^\star[p_0]\ \nabla\phi + \int_{\Gamma_\star} \theta_\star\ \widehat{h[p_0]}\ \phi, \quad \forall\ \phi\in V,$$

with $p_0 \geq 0$, $p_0(1-\Theta)=0$ and $0\leq\Theta\leq 1$ a.e. (by Proposition 2.30), and the proof is concluded. □

Remark 2.34. *An interesting point is to consider that, in the homogenized problem, we are not able to identify an "equivalent gap" since anisotropic effects classically appear in the coefficients. Nevertheless, Theorem 2.24 allows us to define not only one single saturation function Θ but also one single deformation*

$$\int_\Omega k(x,z)\ p_0(z)\ dz,$$

which is an important point in terms of mechanical applications.

2.2.5 Particular cases

In this subsection, due to particular choices of the roughness pattern, local problems have obvious analytical solutions (see [BMV05] or Chapter 1) so that it is possible to obtain self contained Reynolds equations for p_0 and one single saturation function Θ.

Theorem 2.35 (Transverse roughness)**.** *If λ does not depend on y_2, the homogenized problem $(\mathcal{P}_\theta^\star)$ is*

$$\left\{\begin{array}{l} \text{Find } (p_0,\ \Theta)\in V\times L^\infty(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega e^{-\alpha p_0}\ \mathcal{A}[p_0]\cdot\nabla p_0\ \nabla\phi = \int_\Omega \Theta\ b^\star[p_0]\ \frac{\partial\phi}{\partial x_1} + \int_{\Gamma_\star}\theta_\star\ \widehat{h[p_0]}\ \phi, \quad \forall\ \phi\in V \\ p_0\geq 0, \quad p_0\ (1-\Theta)=0, \quad 0\leq\Theta\leq 1, \quad \text{a.e.,} \end{array}\right.$$

with the following homogenized coefficients:

$$\mathcal{A}[p_0](x) = \begin{pmatrix} \dfrac{1}{\widetilde{h^{-3}[p_0]}(x)} & 0 \\ 0 & \widetilde{h^3[p_0]}(x) \end{pmatrix}, \quad b^\star[p_0](x) = \frac{\widetilde{h^{-2}[p_0]}(x)}{\widetilde{h^{-3}[p_0]}(x)}.$$

Moreover $(\mathcal{P}^\star_\theta)$ *admits at least* (p_0, Θ) *as a solution where* (p_0, θ_0) *is the two-scale limit of* $(p_\varepsilon, \theta_\varepsilon)$ *(solution of problem* $(\mathcal{P}^\varepsilon_\theta)$*), and the link between the microscopic / macroscopic saturation function is given by:*

$$\Theta(x) = \Big[\frac{1}{\widetilde{h^{-2}[p_0]}}\ \Big(\widetilde{\frac{\theta_0}{h^2[p_0]}}\Big)\Big](x).$$

Theorem 2.36 (Longitudinal roughness)**.** *If* λ *does not depend on* y_1*, the homogenized problem* $(\mathcal{P}^\star_\theta)$ *is*

$$\begin{cases} \text{Find } (p_0,\ \Theta) \in V \times L^\infty(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega e^{-\alpha p_0}\ \mathcal{A}[p_0] \cdot \nabla p_0\ \nabla \phi = \int_\Omega \Theta\ b^\star[p_0]\ \frac{\partial \phi}{\partial x_1} + \int_{\Gamma_\star} \theta_\star\ \widehat{h[p_0]}\ \phi, \quad \forall\ \phi \in V \\ p_0 \geq 0, \quad p_0\ (1-\Theta) = 0, \quad 0 \leq \Theta \leq 1, \quad \text{a.e.,} \end{cases}$$

with the following homogenized coefficients:

$$\mathcal{A}[p_0](x) = \begin{pmatrix} \widetilde{h^3[p_0]}(x) & 0 \\ 0 & \dfrac{1}{\widetilde{h^{-3}[p_0]}(x)} \end{pmatrix}, \quad b^\star[p_0](x) = \widetilde{h[p_0]}(x).$$

Moreover $(\mathcal{P}^\star_\theta)$ *admits at least* (p_0, Θ) *as a solution where* (p_0, θ_0) *is the two-scale limit of* $(p_\varepsilon, \theta_\varepsilon)$ *(solution of problem* $(\mathcal{P}^\varepsilon_\theta)$*), and the link between the microscopic / macroscopic saturation function is given by:*

$$\Theta(x) = \frac{\widetilde{(\theta_0\ h[p_0])}}{\widetilde{h[p_0]}}(x).$$

In the next section, we focus on numerical tests which illustrate the theoretical results which have been established in this section. In particular, we are interested in longitudinal and transverse roughness cases which have a great interest in terms of mechanical applications.

2.3 Numerical examples

In this section, the numerical simulation of micro-elastohydrodynamic contacts is performed to illustrate the theoretical results of convergence stated in the previous sections. For the numerical solution of the ε dependent problems and their corresponding homogenized one, we propose an algorithm based on a fixed-point iteration between the hydrodynamic (Elrod-Adams) problem and the elastic (Hertz) one [DGV96b]. Furthermore, the hydrodynamic problem is solved using the characteristics method to deal with the convection term combined with a finite element spatial discretization and a duality method for the maximal monotone nonlinearity associated to the Elrod-Adams model. The elastic problem is approximated by using appropriate quadrature formulas in Equation (2.3). More precisely, the triangle edges midpoints are chosen as integration nodes. The combination of these numerical techniques has been already successfully used in previous papers dealing with the elastohydrodynamic related smooth problems [DGV96b, DGV02].

We adress the numerical simulation of dimensionless ball bearing contacts so that, for a domain $\Omega =]-2l_1, l_1[\times]-l_2, l_2[$, we pose the problem

$$\begin{cases} \text{Find } (p_\varepsilon, \theta_\varepsilon) \in V \times L^\infty(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega h_\varepsilon^3[p_\varepsilon]\ e^{-\alpha p_\varepsilon}\ \nabla p_\varepsilon\ \nabla\phi = \int_\Omega \theta_\varepsilon\ h_\varepsilon[p_\varepsilon]\ \frac{\partial\phi}{\partial x_1} + \int_{\Gamma_\star} \theta_\star\ h_\varepsilon[p_\varepsilon]\ \phi, \quad \forall\ \phi \in V \\ p_\varepsilon \geq 0, \quad p_\varepsilon\ (1-\theta_\varepsilon) = 0, \quad 0 \leq \theta_\varepsilon \leq 1, \quad \text{a.e.,} \end{cases}$$

where the effective gap can be written as $h_\varepsilon[p_\varepsilon](x) = h_r^\varepsilon(x) + h_d[p_\varepsilon](x)$, the rigid and elastic contributions being given by

$$\begin{aligned} h_r^\varepsilon(x) &= h_0 + \frac{x_1^2 + x_2^2}{2} + \alpha_1 \sin\left(\frac{6l_1\pi(x_1+2l_1)}{\varepsilon}\right) + \alpha_2 \sin\left(\frac{4l_2\pi(x_2+l_2)}{\varepsilon}\right), \\ h_d[p_\varepsilon](x) &= \frac{2}{\pi^2}\int \frac{p_\varepsilon(z)}{\sqrt{(x_1-z_1)^2 + (x_2-z_2)^2}}\, dz \end{aligned}$$

and the following set of parameters:

- transverse roughness : $l_1 = 2, \quad l_2 = 5, \quad h_0 = 0.5, \quad (\alpha_1, \alpha_2) = (0.5h_0, 0)$,
- longitudinal roughness : $l_1 = 2, \quad l_2 = 2, \quad h_0 = 0.6, \quad (\alpha_1, \alpha_2) = (0, 0.85h_0)$.

Other parameters involved in the equation are taken to $\alpha = 1$ (piezoviscosity parameter) and $\theta_\star = 0.3$ (boundary data). The previous data have been taken from dimensionless equations associated to a small load imposed problem [DGV02].

The homogenized problem, in the transverse or longitudinal case, can be written as

$$\begin{cases} \text{Find } (p_0,\ \Theta) \in V \times L^\infty(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega e^{-\alpha p_0} \begin{pmatrix} a_1[p_0] & 0 \\ 0 & a_2[p_0] \end{pmatrix} \cdot \nabla p_0\ \nabla\phi = \int_\Omega \Theta\ b[p_0]\ \frac{\partial\phi}{\partial x_1} + \int_{\Gamma_\star} \theta_\star c[p_0]\ \phi,\ \forall\ \phi \in V \\ p_0 \geq 0, \quad p_0\ \big(1-\Theta\big) = 0, \quad 0 \leq \Theta \leq 1, \quad \text{a.e.} \end{cases}$$

In Table 2.1, we present the functional coefficients a_1, a_2, b and c that appear in the homogenized problem for purely transverse and purely longitudinal roughness cases which have been partially computed with MATHEMATICA Software Package.

	Transverse roughness	Longitudinal roughness
$h[p](x,y)$	$h_r(x) + h_d[p](x) + \alpha_1 \sin\left(2\pi y_1\right)$	$h_r(x) + h_d[p](x) + \alpha_2 \sin\left(2\pi y_2\right)$
$a_1[p_0]$	$2\dfrac{\left((h_r + h_d[p])^2 - \alpha_1^2\right)^{5/2}}{2(h_r + h_d[p])^2 + h_r^2}$	$(h_r + h_d[p])^3 + \dfrac{3}{2}\,(h_r + h_d[p])\,\alpha_2^2$
$a_2[p_0]$	$(h_r + h_d[p])^3 + \dfrac{3}{2}\,(h_r + h_d[p])\,\alpha_1^2$	$2\,\dfrac{\left((h_r + h_d[p])^2 - \alpha_2^2\right)^{5/2}}{2(h_r + h_d[p])^2 + \alpha_2^2}$
$b[p_0]$	$2(h_r + h_d[p])\,\dfrac{(h_r + h_d[p])^2 - \alpha_1^2}{2(h_r + h_d[p])^2 + \alpha_1^2}$	$h_r + h_d[p]$
$c[p_0]$	$h_r + h_d[p]$	$h_r + h_d[p]$

Table 2.1. Elastohydrodynamic homogenized coefficients

Let us remark that the smooth rigid gap is given by

$$h_r(x) = h_0 + \frac{x_1^2 + x_2^2}{2}.$$

2.3.1 Case 1: transverse roughness tests

Although numerical tests have been performed for different spatial meshes in order to validate the convergence of the methods, we just present the results corresponding to $\Delta x_1 = 0.02$ and $\Delta x_2 = 0.133$ so that we have 44400 triangles and 22575 vertices. Furthermore, the artificial time step in the characteristics method [BCV98, DGV96b], $\Delta t = \Delta x_1$; the Bermudez-Moreno parameters are $\omega = 1$ and $\lambda = 1/(2\omega)$; the stopping test in all algorithms is equal to $\delta = 10^{-5}$ (corresponding to the relative error in the discrete L^2 norm between two iterations).

The computer results illustrate the convergences stated in previous sections. First in FIG.2.1, we show the strong convergence of the pressure to the homogenized one, when ε tends to 0, by plotting the cuts at $x_2^0 = 0$, for different values of ε and the homogenized solution. In FIG.2.2, the homogenized pressure over the whole domain is presented. In FIG.2.3 and 2.4, analogous plots for the deformation are displayed to illustrate the convergence and the homogenized distribution. Finally, in FIG.2.5 and 2.6, the results for the saturation are shown. To be noticed is the weak convergence of the saturation, linked to the existence of oscillations which are not damped, unlike the pressure and deformation. We can also notice that the deformation oscillations are damped very easily (when compared to the pressure oscillations). This is due to the regularizing effects of the Hertz kernel (which is a convolution-like operator).

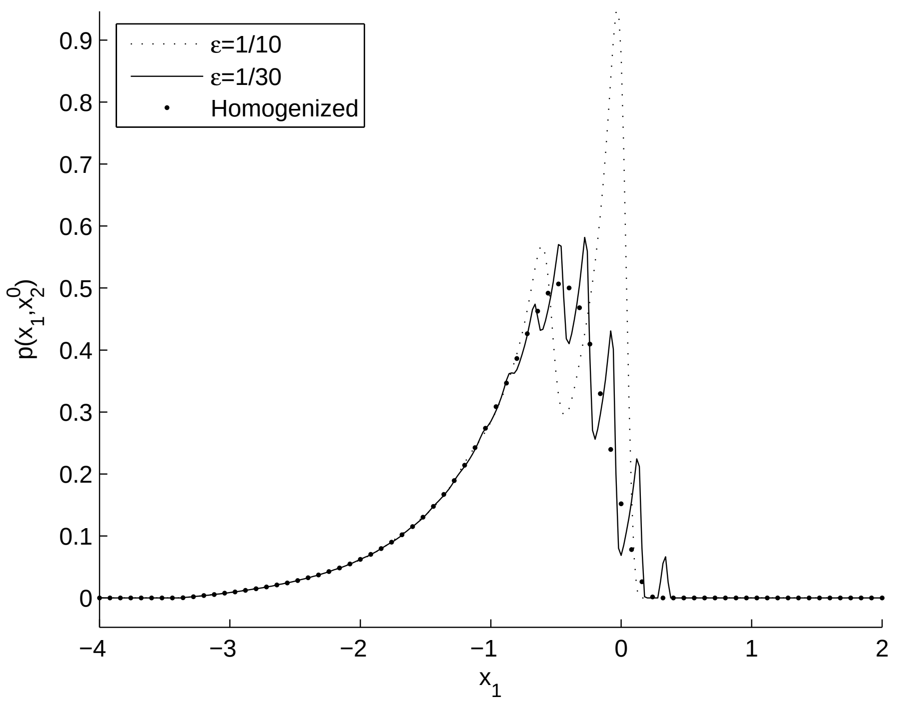


Figure 2.1. Elastohydrodynamic pressure at $x_2^0 = 0$ (transverse roughness)

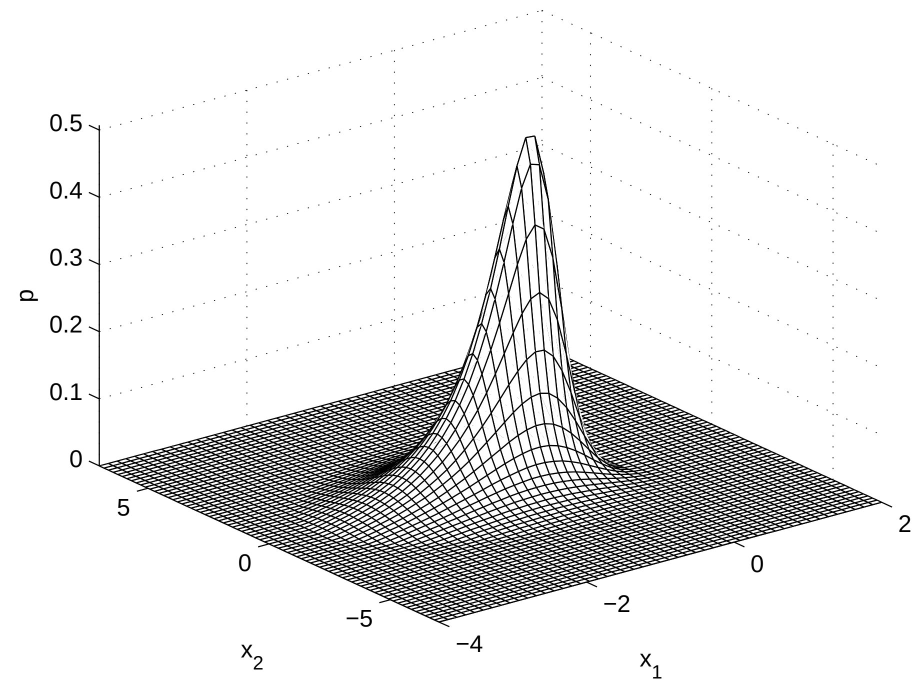


Figure 2.2. Elastohydrodynamic homogenized pressure (transverse roughness)

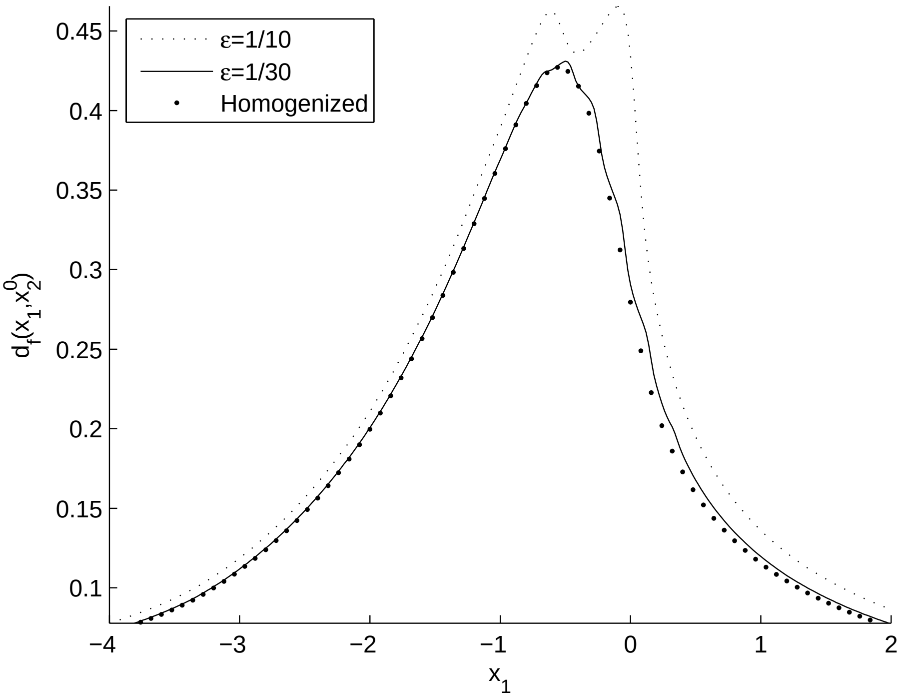


Figure 2.3. Elastohydrodynamic deformation at $x_2^0 = 0$ (transverse roughness)

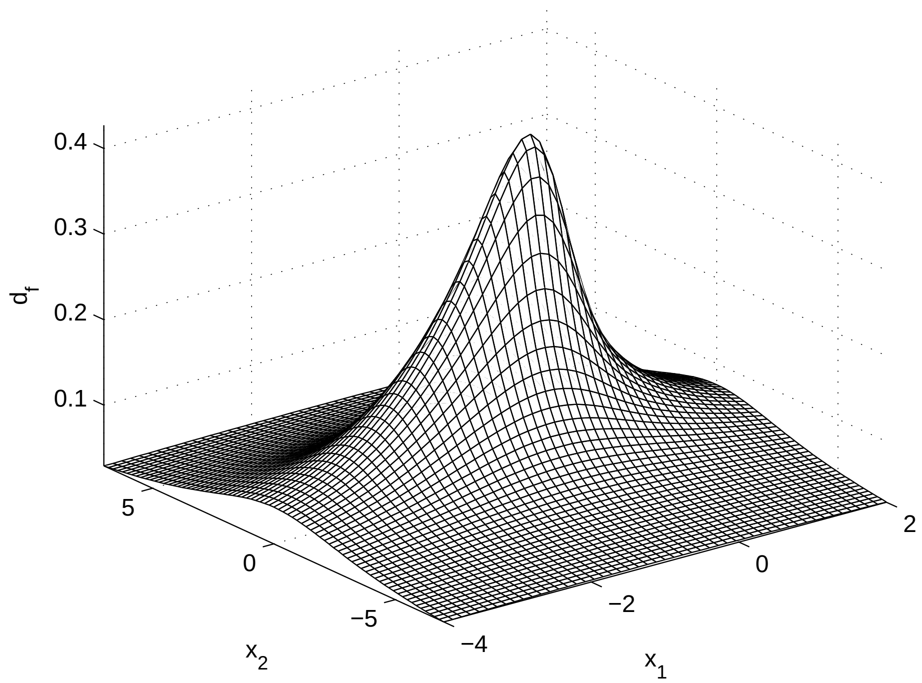


Figure 2.4. Elastohydrodynamic homogenized pressure (transverse roughness)

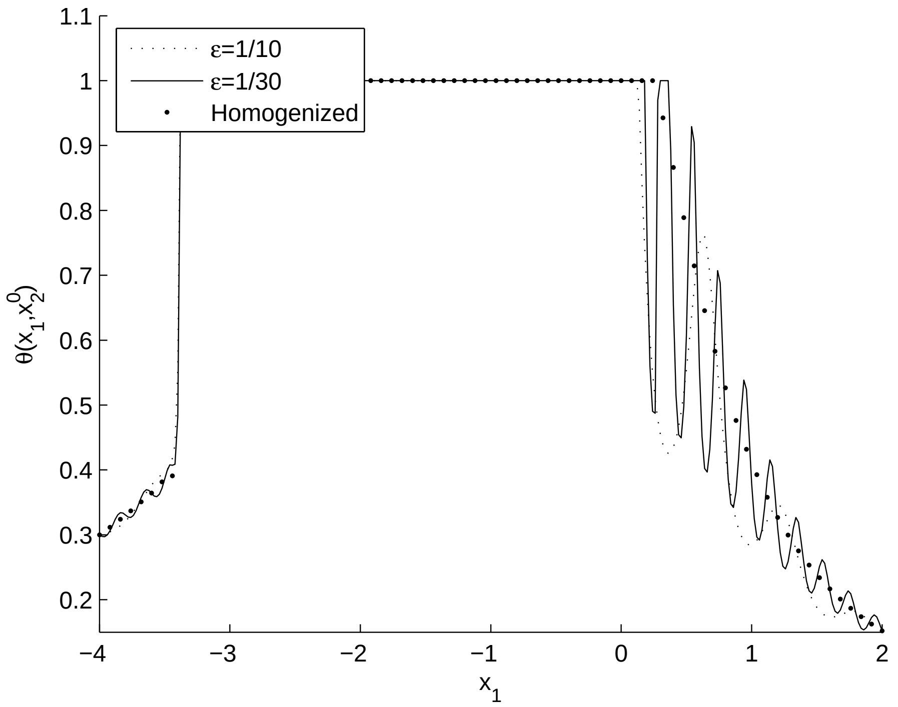


Figure 2.5. Elastohydrodynamic saturation at $x_2^0 = 0$ (transverse roughness)

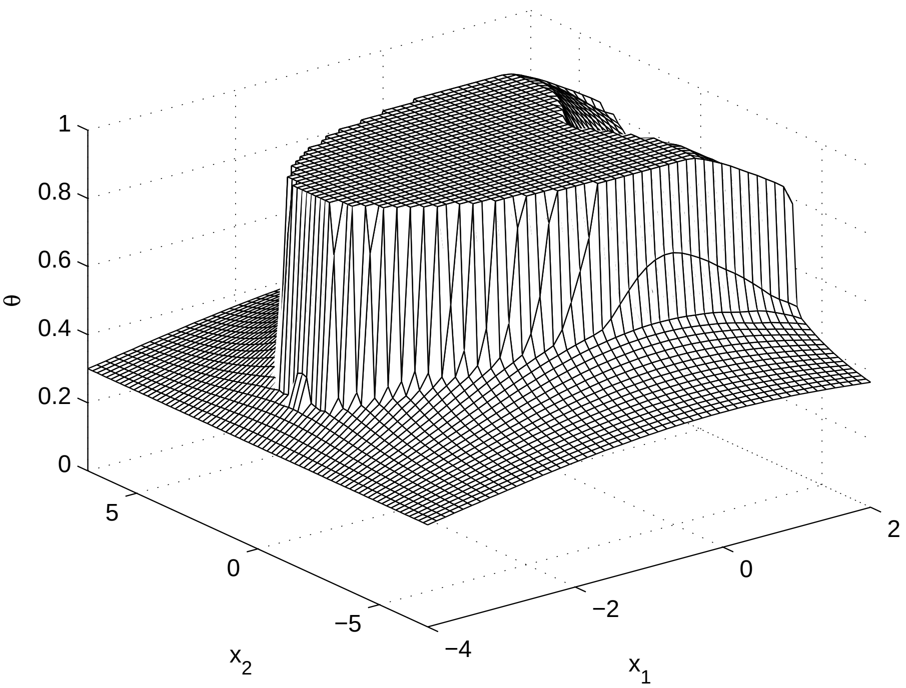


Figure 2.6. Elastohydrodynamic homogenized saturation (transverse roughness)

2.3.2 Case 2: longitudinal roughness tests

In this case, we present the results corresponding to $\Delta x_1 = 0.06$ and $\Delta x_2 = 0.02$ so that we have 40000 triangles and 20301 vertices.

The computer results illustrate the convergences stated in previous sections. First in Fig.2.7, we show the strong convergence of the pressure to the homogenized one, when ε tends to 0, by plotting the cuts at $x_1^0 = -2.5$ (corresponding to the maximum homogenized pressure), for different values of ε and the homogenized solution. In Fig.2.8, the homogenized pressure over the whole domain is presented. In Fig.2.9 and 2.10, the homogenized deformation and saturation over the whole domain are presented. Notice that at $x_1^0 = -2.5$, all saturations are identically 1.

2.3.3 Influence of the elastic contribution over the roughness effects

We compare the results, in a transverse roughness case, between hydrodynamic and elastohydrodynamic configurations.

In Fig.2.11 and 2.12, we present the pressure and saturation profiles in the following cases: $l_1 = 2$, $l_2 = 2$, $h_0 = 0.6$ and

- hydrodynamic smooth solution : $\alpha_1 = 0,\quad \alpha_2 = 0,\quad k \equiv 0,$
- hydrodynamic homogenized solution : $\alpha_1 = 0.85\ h_0,\quad \alpha_2 = 0,\quad k \equiv 0,$
- elastohydrodynamic smooth solution : $\alpha_1 = 0,\quad \alpha_2 = 0,\quad k \neq 0,$
- elastohydrodynamic homogenized solution : $\alpha_1 = 0.85\ h_0,\quad \alpha_2 = 0,\quad k \neq 0.$

The figures correspond to a fixed $x_2^0 = 0$. It allows us to see the influence of the roughness over the pressure distribution in each case (hydrodynamic or elastohydrodynamic), but also the importance of the roughness over the maximum pressure. The roughness effects are clearly more important in the hydrodynamic regime than in the elastohydrodynamic one.

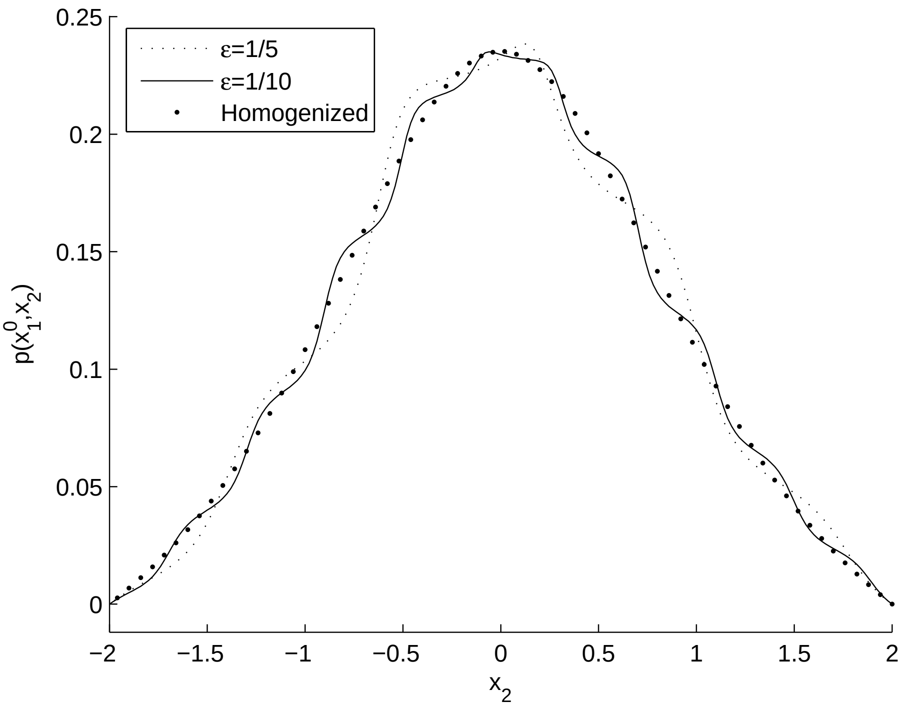


Figure 2.7. Elastohydrodynamic pressure at $x_1^0 = -2.5$ (longitudinal roughness)

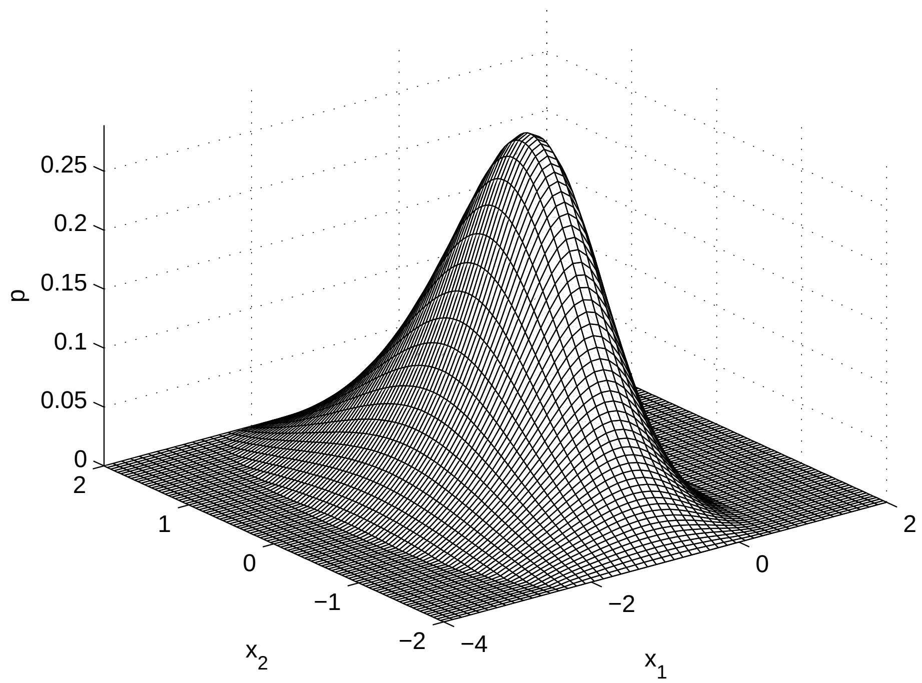


Figure 2.8. Elastohydrodynamic homogenized pressure (longitudinal roughness)

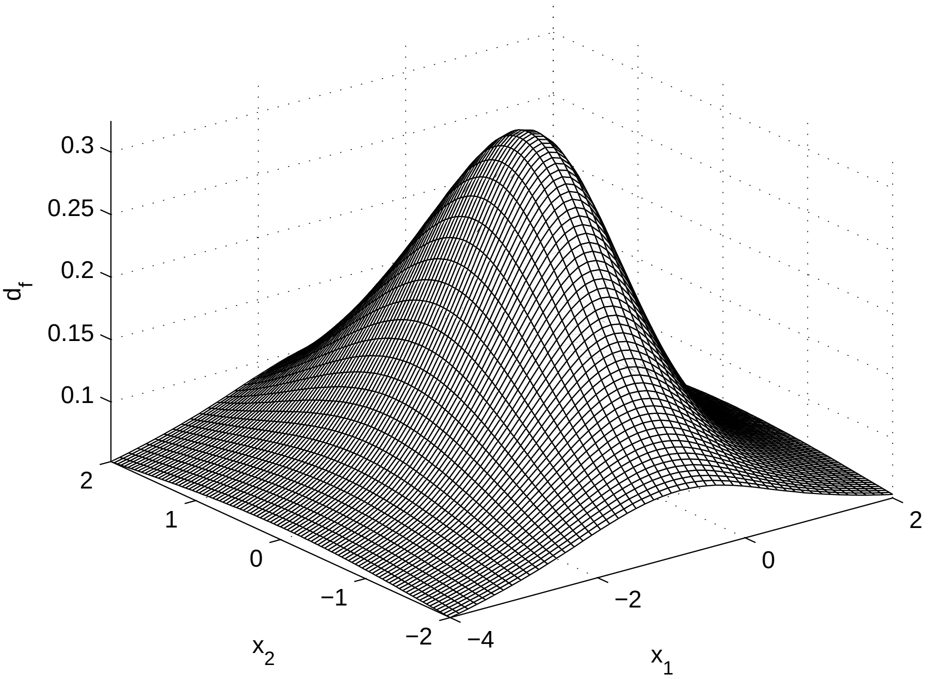


Figure 2.9. Elastohydrodynamic homogenized deformation (longitudinal roughness)

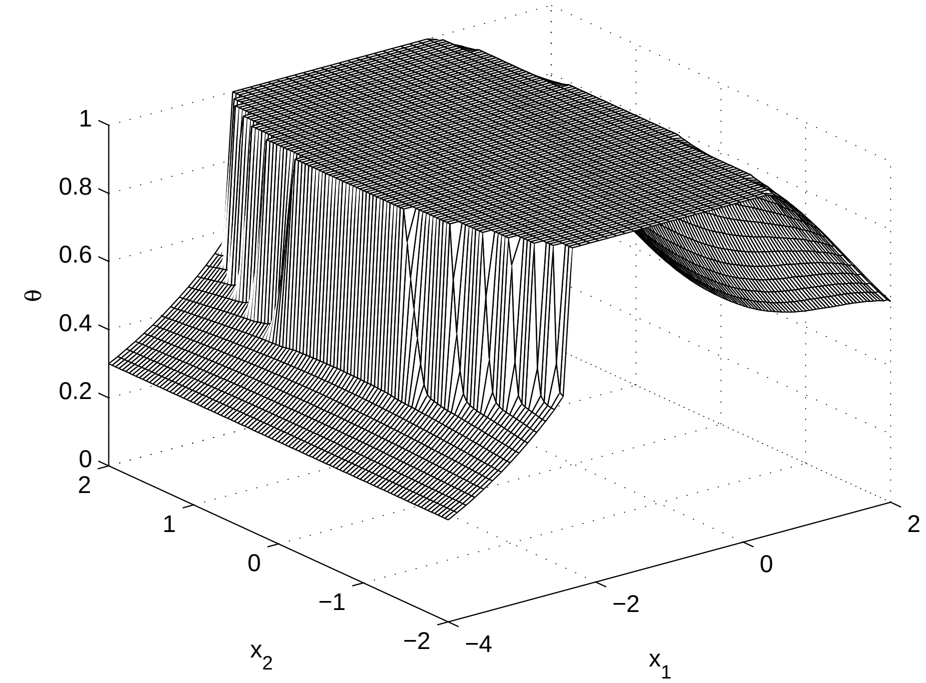


Figure 2.10. Elastohydrodynamic homogenized saturation (longitudinal roughness)

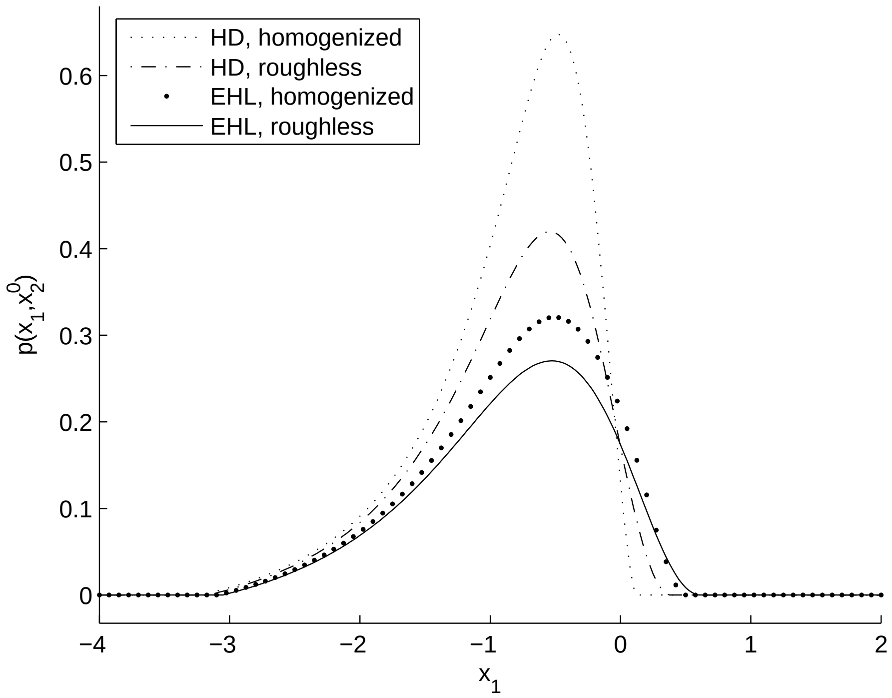


Figure 2.11. Roughness effects over EHL and hydrodynamic pressures

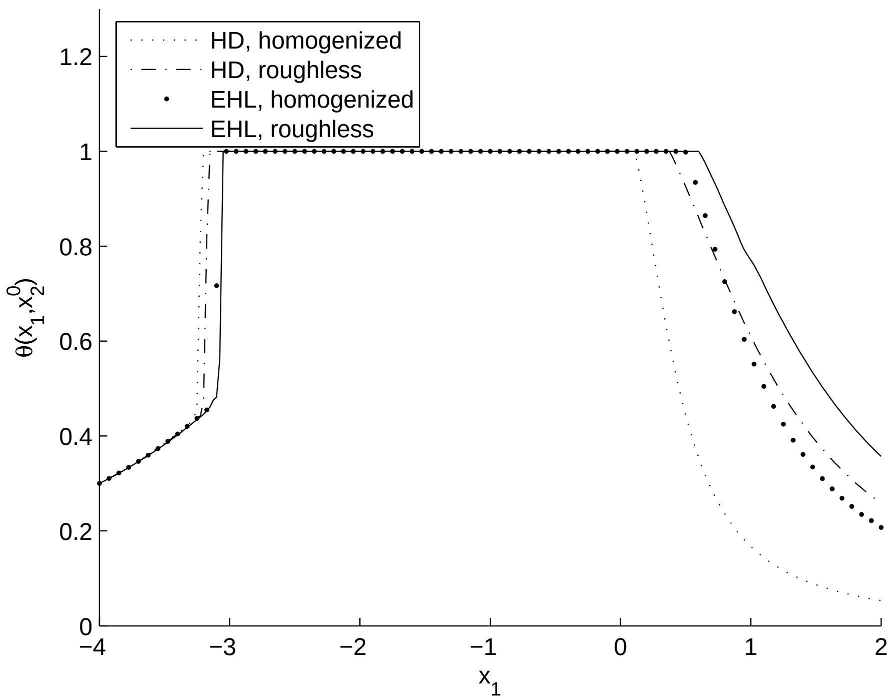


Figure 2.12. Roughness effects over EHL and hydrodynamic saturations

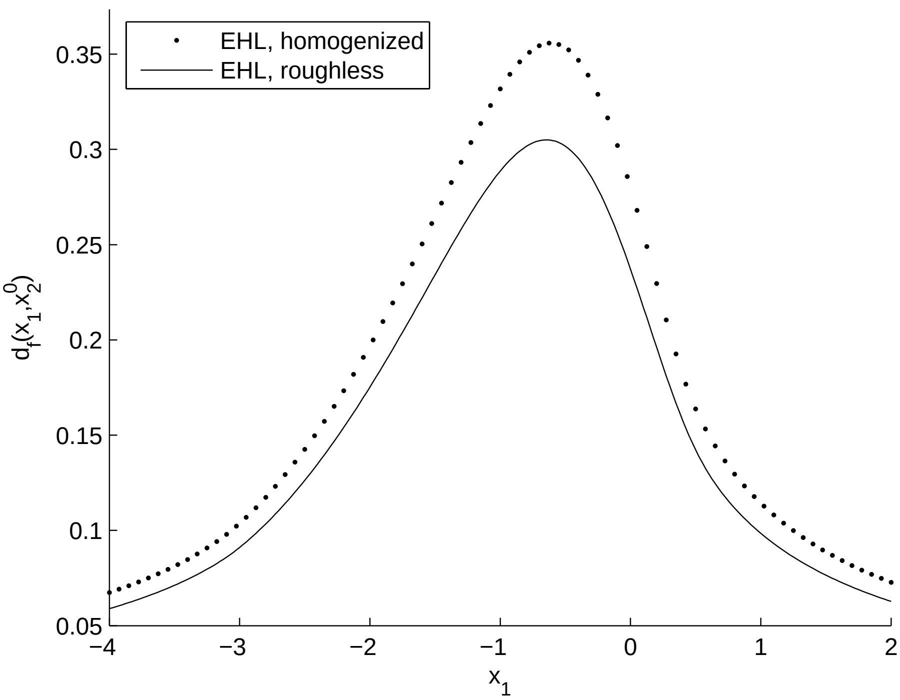


Figure 2.13. Influence of the rougness effects on the deformation

Bibliography

[All92] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.

[Bar93] C. Barus. Isothermals, isopiestics and isometrics relative to viscosity. *American Journal of Science*, 45:87–96, 1893.

[BC86a] G. Bayada and M. Chambat. Sur quelques modélisations de la zone de cavitation en lubrification hydrodynamique. *J. Méc. Théor. Appl.*, 5(5):703–729, 1986.

[BC86b] G. Bayada and M. Chambat. The transition between the Stokes equations and the Reynolds equation: a mathematical proof. *Appl. Math. Optim.*, 14(1):73–93, 1986.

[BCV98] G. Bayada, M. Chambat, and C. Vázquez. Characteristics method for the formulation and computation of a free boundary cavitation problem. *J. Comput. Appl. Math.*, 98(2):191–212, 1998.

[BCV03] G. Bayada, B. Cid, and C. Vázquez. Two-scale homogenization study of a Reynolds-rod elastohydrodynamic model. *Math. Models Methods Appl. Sci.*, 13(2):259–293, 2003.

[BEATV96] G. Bayada, M. El Alaoui Talibi, and C. Vázquez. Existence of solutions for elastohydrodynamic piezoviscous lubrication problems with a new model of cavitation. *European J. Appl. Math.*, 7(1):63–73, 1996.

[BF89] G. Bayada and J.-B Faure. A double-scale analysis approach of the Reynolds roughness. Comments and application to the journal bearing. *ASME J. Tribol.*, 111:323–330, 1989.

[BMV05] G. Bayada, S. Martin, and C. Vázquez. Two-scale homogenization of a hydrodynamic Elrod-Adams model. *Asymptot. Anal.*, 44(1-2):75–110, 2005.

[Bre83] H. Brezis. *Analyse fonctionnelle.* Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[CDG02] D. Cioranescu, A. Damlamian, and G. Griso. Periodic unfolding and homogenization. *C. R. Math. Acad. Sci. Paris*, 335(1):99–104, 2002.

[DGV96a] J. Durany, G. García, and C. Vázquez. A mixed Dirichlet-Neumann problem for a nonlinear Reynolds equation in elastohydrodynamic piezoviscous lubrication. *Proc. Edinburgh Math. Soc. (2)*, 39(1):151–162, 1996.

[DGV96b] J. Durany, G. García, and C. Vázquez. Numerical computation of free boundary problems in elastohydrodynamic lubrication. *Appl. Math. Modelling*, 20:104–113, 1996.

[DGV02] J. Durany, G. García, and C. Vázquez. Numerical simulation of a lubricated Hertzian contact problem under imposed load. *Finite Elem. Anal. Des.*, 38(7):645–658, 2002.

[DH77] D. Dowson and G. R. Higginson. *Elastohydrodynamic lubrication.* Pergamon Press, Oxford, Great Britain, 1977.

[EA75] H. G. Elrod and M. L. Adams. A computer program for cavitation. *Cavitation and related phenomena in lubrication - Proceedings - Mech. Eng. Publ. ltd*, pages 37–42, 1975.

[GN94] D. Goeleven and V. H. Nguyen. On the one-dimensional nonlinear elastohydrodynamic lubrication. *Bull. Austral. Math. Soc.*, 50(3):353–372, 1994.

[Hoo98] C. J. Hooke. The behaviour of low-amplitude surface roughness under line contacts. *Proc. Instn. Mech. Engrs.*, 213:275–285, 1998.

[Hu90] B. Hu. A quasi-variational inequality arising in elastohydrodynamics. *SIAM J. Math. Anal.*, 21(1):18–36, 1990.

[Jai95] M. Jai. Homogenization and two-scale convergence of the compressible Reynolds lubrication equation modelling the flying characteristics of a rough magnetic head over a rough rigid-disk surface. *RAIRO Modél. Math. Anal. Numér.*, 29(2):199–233, 1995.

[LNW02] D. Lukkassen, G. Nguetseng, and P. Wall. Two-scale convergence. *Int. J. Pure Appl. Math.*, 2(1):35–86, 2002.

[Ngu89] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20(3):608–623, 1989.

[OW85] J. T. Oden and S. R. Wu. Existence of solutions to the Reynolds' equation of elastohydrodynamic lubrication. *Internat. J. Engrg. Sci.*, 23(2):207–215, 1985.

[Rey86] O. Reynolds. On the theory of lubrication and its application to Mr Beauchamp tower's experiments, including an experimental determination of the viscosity of olive oil. *Phil. Trans. Roy. Soc.*, A 117:157–234, 1886.

[Rod93] J.-F. Rodrigues. Remarks on the Reynolds problem of elastohydrodynamic lubrication. *European J. Appl. Math.*, 4(1):83–96, 1993.

3

First order quasilinear equations with boundary conditions in the L^∞ framework

Article soumis pour publication.

Abstract *In this chapter, we study a class of first order quasilinear equations on bounded domains in the L^∞ framework. Using the "semi Kružkov entropy-flux pairs", we define a weak-entropy solution, state an existence and uniqueness result, and a set preserving result.*

3.0 Introduction

In this chapter, $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded smooth domain. Let us denote by $\partial\Omega$ the boundary of Ω and by n the outer normal vector to $\partial\Omega$. We denote $Q_T \equiv (0,T) \times \Omega$ and $\Sigma_T \equiv (0,T) \times \partial\Omega$. Let us consider a scalar conservation law:

$$\frac{\partial u}{\partial t} + \nabla \cdot \Big(f(t,x,u) \Big) + g(t,x,u) = 0, \quad \text{on } Q_T, \tag{3.1}$$

$$u(0,\cdot) = u^0, \quad \text{on } \Omega, \tag{3.2}$$

$$\text{“}u = u^D\text{”}, \quad \text{on } \Sigma_T, \tag{3.3}$$

where the sense of the boundary condition will be precised further. We consider the following assumption:

Assumption 13.

(i) *f and g are two functions defined on* $[0,T]\times\overline{\Omega}\times\mathbb{R}$ *such that*

$$f \in \left(C^2([0,T]\times\overline{\Omega}\times[a,b])\right)^d, \quad g \in C^2([0,T]\times\overline{\Omega}\times[a,b]),$$

(ii) *f, $\nabla\cdot f$ and g are Lipschitz continuous with respect to u, uniformly in (t,x), the constants of Lipschitz continuity being respectively denoted $\mathcal{L}_{[f]}$, $\mathcal{L}_{[\nabla\cdot f]}$, $\mathcal{L}_{[g]}$,*

(iii) $(u^0,u^D)\in L^\infty(\Omega;[a,b])\times L^\infty(\Sigma_T;[a,b])$,

(iv) $(\nabla\cdot f+g)(\cdot,\cdot,a)\le 0$ *and* $(\nabla\cdot f+g)(\cdot,\cdot,b)\ge 0$ *uniformly in* (t,x).

Due to the presence of a non-autonomous flux f or source term g, interest in Equations (3.1)–(3.3) may be related to many problems applied to many mathematical and physical areas: oil engineering (two-phase flows in porous media [CJ86, GMT96, AS79, Kaa99]), lubrication theory (two-phase flow in a thin domain [Pao03], modelling of cavitation phenomena [BMV05]), environmental sciences (pollutant transport in manure-like fields [WPW$^+$01]), chemical engineering and wastewater treatment (sedimentation of flocculated suspensions in clarifier-thickener units [BKT05]). It is of course an important feature to analyse the related phenomena in order to guarantee the well-posedness of the problem and also qualitative properties of the (possible) solution. Let us introduce in details some physical problems which fall into the scope of our study:

- **Lubrication theory -** The flow of two miscible fluids in a thin film has been derived by Sabil and Paoli in [Pao03] and further detailed by Bayada, Martin and Vázquez in [BMV05] (see Chapter 4). Thus, it allows the modelling of a multifluid flow in a one dimensional lubricated device as a slider or journal bearing, for instance. The behaviour of the saturation s of the reference fluid obeys the following law:

$$h(x)\frac{\partial s}{\partial t}+\frac{\partial}{\partial x}\Big(Q_{in}(t)\ f_1(s)+v_0\ h(x)\ f_2(s)\Big)=0, \quad (t,x)\in(0,T)\times(0,L). \tag{3.4}$$

Here, L is the length of the device, h is the normalized gap between the two surfaces in relative motion, v_0 is the shear velocity of the lower surface (the upper one being fixed, for instance) and $Q_{in}(t)$ is the flow input. Moreover, the auxiliary flux functions f_1 and f_2 have a particular shape: $f_1(0)=0$, $f_1(1)=1$ (typically, f_1 is S-shaped on $[0,1]$) and $f_2(0)=f_2(1)=0$. Equation (3.4) is completed with initial and boundary conditions s^0 and s^D satisfying, for obvious physical relevance,

$$\left[\min\left(\inf_{\Omega} s^0, \inf_{\Sigma_T} s^D\right), \max\left(\sup_{\Omega} s^0, \sup_{\Sigma_T} s^D\right)\right]\subset[0,1].$$

Using an appropriate change of variables $(t, x) \mapsto (\mathcal{T}, Y)$, it has been proven in [BMV05] (see Chapter 4) that the problem can be reduced to a scalar conservation law with respect to the function u defined by $u(\mathcal{T}(t), Y(x)) = s(t, x)$, namely

$$\frac{\partial u}{\partial \tau} + \frac{\partial}{\partial y}\Big(f_1(u) + v_0\,\frac{h \circ Y^{-1}(y)}{Q_{in}(\mathcal{T}^{-1}(\tau))}\;f_2(u)\Big) = 0, \quad (\tau, y) \in (0, \widetilde{T}) \times (0,1). \tag{3.5}$$

with corresponding initial and boundary conditions u^0 and u^D defined by $u^0(y) = s^0(Y^{-1}(y))$ and $u^D(\tau, y) = s^D(\mathcal{T}^{-1}(\tau), Y^{-1}(y))$. Equation (3.5) is obviously a scalar conservation law with respect to a saturation function u, unlike Equation (3.4). Notice that the non-autonomous property of the flux function is induced by the shear effects ($v_0 \neq 0$) and spatial variations of h (typically, h has a converging-diverging profile). Moreover, the boundary conditions may lack regularity (typically, $s^D \in L^\infty((0,T) \times \partial]0, L[))$, due to fast changes of the supply regime. For physical relevance, it is an important feature to state that Equation (3.4) (or, equivalently, Equation (3.5)), with the corresponding initial and boundary conditions, admits a (unique) solution, in a sense that has to be precised, and that it takes its values in the set $[0, 1]$.

- **Environmental sciences -** The modelling of pollutant transport taking into account a surface source during rainfall-runoff is described by Walter, Parlange, Walter, Xin and Scott in [WPW+01]. It allows to model some step of the pollution process due to the runoff from manure spread fields, an important mode of non-point source pollution. Actually, pollutant release involves two processes, horizontal convection (which occurs in the bottom region of the source) and vertical convective diffusion and/or dispersion from the upper region. The pollutant transport mechanisms for the bottom region and upper regions of the source are modelled as independent processes[1]. The convective process is described by the following transport equation including a source term

$$\frac{\partial c}{\partial t} + \nabla \cdot \Big(U\ c\Big) + \frac{1}{h}(ic - J_b) = 0, \tag{3.6}$$

where c denotes the concentration of pollutant, U the convective velocity of the flow, h the depth of flow, i the effective rainfall intensity and J_b the rate of solute uptake from the source into the flow. Here, for the sake of simplicity, we suppose that h is constant and U satisfies $\nabla \cdot U = 0$. However, this assumption may be

[1]The justification for the uncoupling of these processes lies in the assumption that the time required to flush out contaminates from the lower region is much shorter than the time required to flush out contaminates from the upper region

relaxed[2]. Moreover, Equation (3.6) is completed with some initial condition $c^0 \equiv 0$ and an homogeneous boundary condition $c^D \equiv 0$ on the boundary part for which $U \cdot n < 0$. The analysis of this set of equations allows to ensure the well-posedness of the physical problem and guarantees the boundedness of the concentration (see Remark 3.16).

From a mathematical point of view, numerous works have approached or investigated this field. On unbounded domains, the existence and uniqueness of a solution for quasilinear first order equations domains has been solved in the pioneering works of Oleĭnik [Ole59], Volpert [Vol67] and Kružkov [Kru70] who introduced the concept of weak entropy solutions and related "Kružkov entropy-flux pairs". When dealing with bounded domains, under some regularity assumptions on the data, Bardos, Le Roux and Nédélec [BLRN79] also proved existence and uniqueness of a weak entropy solution satisfying a "Kružkov entropy-flux pair" formulation including boundary terms; for this, they introduced an appropriate mathematical boundary condition that must be understood in a particular way. Nevertheless, when considering L^∞ data, the lack of regularity prevents from using the result of Bardos, Le Roux and Nédélec. This difficulty was overcome, at least in the case of autonomous scalar conservation laws on bounded domains, by Otto [Ott96, MNRR96] who introduced "boundary entropy-flux pairs" which enable to state existence and uniqueness of a so-called weak entropy solution and a set preserving result for this solution. Finally, using a lemma proposed by Vovelle [Vov02], it appears that a formulation using "semi Kružkov entropy-flux pairs" is equivalent to a formulation based on "boundary entropy-flux pairs". This "semi Kružkov entropy-flux pairs" formulation is very similar to the initial one of Kružkov, and uses simple algebraic expressions. Now, let us consider the following questions:

- What is the appropriate definition of a weak entropy solution for first order quasilinear equations (i.e. including non-autonomous fluxes and source terms) on bounded domains with L^∞ data ? Answering this question would draw a complete parallel with the results of Bardos, Le Roux and Nédélec [BLRN79] and those of Otto [Ott96] and Vovelle [Vov02]: indeed, the analysis of scalar conservation laws with L^∞ data, initiated by Otto, would be extended to first order quasilinear equations, studied by Bardos, Le Roux and Nédélec.

- What sufficient conditions lead to a set preserving result ? Indeed, such a property is crucial when studying some physical problems: for instance, the saturation of a

[2] The more general case with spatially varying functions h and U (with $q = hU$ and $\nabla \cdot q = 0$) can be considered, leading to a conservation law with respect to hc. Then, as in the lubrication problem, the equation can be reduced to a conservation law with respect to a concentration function.

fluid in porous phase flows may obey the Buckley-Leverett equation; in that case, it is an important feature to prove that the solution lies in $[0,1]$ with similar data.

Thus, it is the purpose of this chapter to give a general framework which is valid for a wide class of quasilinear first order equations on bounded domains with L^∞ data. Among the difficulties, we can observe that, when dealing with non autonomous fluxes and source terms, a "boundary entropy-flux pairs" formulation is not possible anymore. Fortunately, we will see that the concept of "semi Kružkov entropy-flux pairs" allows to overcome difficulties. Moreover, a set preserving result is stated, up to some additional assumptions (Assumption 13 (iv), in particular). As a concluding remark of the chapter, we shall apply the stated results to the physical problems that we have introduced, i.e. lubrication problem and pollutant transport.

Let us recall some fundamental results that fall into the scope of first order quasilinear equations on bounded domains:

Definition-Theorem 1 (Bardos, Le Roux and Nédélec [BLRN79]). *Suppose that*

(i') $f \in \left(C^2((0,T)\times\overline{\Omega}\times\mathbb{R})\right)^d$, $g \in (C^2((0,T)\times\overline{\Omega}\times\mathbb{R})$,

(ii') f, $\nabla\cdot f$ *and* g *are Lipschitz continuous with respect to* u*, uniformly in* (t,x),

(iii') $u^0 \in BV(\Omega)$, $u^D \in C^2(\Sigma_T)$.

There exists a unique $u \in BV(Q_T)$ *(which is called the weak entropy solution of problem (3.1)–(3.3)) satisfying*

$$(\mathcal{P}_0)\begin{cases} \displaystyle\int_{Q_T}\Bigg\{\Big|u-k\Big|\frac{\partial\varphi}{\partial t} + (\operatorname{sgn}(u-k)(f(t,x,u)) - f(t,x,k)))\ \nabla\varphi \\ \qquad - \operatorname{sgn}(u-k)\left(\nabla\cdot f(t,x,k) + g(t,x,u)\right)\ \varphi\Bigg\}\ dx\ dt \\ \qquad + \displaystyle\int_\Omega \Big|u^0-k\Big|\ \varphi(0,x)\ dx \\ \qquad - \displaystyle\int_{\Sigma_T}\operatorname{sgn}(u^D-k)\Bigg\{f(t,r,\gamma u) - f(t,r,k)\Bigg\}\cdot n(r)\ \varphi\ d\gamma(r)\ dt \geq 0, \\ \forall\phi\in\mathcal{D}((-\infty,T)\times\mathbb{R}^d),\ \phi\geq 0,\ \forall k\in\mathbb{R}. \end{cases}$$

Here γu *denotes the trace of* u *on* $\partial\Omega$.

The key point of this paper was to introduce "Kružkov entropy-flux pairs", namely the pair:

$$\Big(\Big|u-k\Big|, \operatorname{sgn}(u-k)(f(t,x,u)-f(t,x,k))\Big)$$

along with the so-called "BLN" condition which allows to give a rigorous meaning to the way the boundary condition is satisfied. Nevertheless, when looking for solutions with less regularity, the notion of trace on the boundary is not relevant anymore. In the L^∞ framework, the notion of weak entropy solution is essentially due to Otto [Ott96] who introduced the so-called "boundary entropy-flux pairs"

$$\Big(H(u,k), Q_{[f]}(u,k)\Big)$$

in order to study autonomous scalar conservation laws on bounded domains with L^∞ data. This formulation allows to recover the "BLN" condition and ensures existence and uniqueness of a weak entropy solution:

Definition-Theorem 2 (Otto [Ott96])**.** *Suppose that*

(i") $f \in \big(C^2(\mathbb{R})\big)^d$ (i.e. the flux is autonomous, only depends on the variable u), $g \equiv 0$,

(ii") f is Lipschitz continuous,

(iii") $(u^0, u^D) \in L^\infty(\Omega; [a,b]) \times L^\infty(\Sigma_T; [a,b])$.

Let $(H, Q_{[f]})$ be in $C^1(\mathbb{R}^2) \times \big(C^1(\mathbb{R}^2)\big)^d$. The pair $(H, Q_{[f]})$ is said to be a "boundary entropy-flux pair" (for the flux f) if:

1. *for all $w \in \mathbb{R}$, $s \mapsto H(s,w)$ is a convex function,*
2. *$\forall w \in \mathbb{R}$, $\partial_s Q_{[f]}(s,w) = \partial_1\ H(s,w) f'(s)$,*
3. *$\forall w \in \mathbb{R}$, $H(w,w) = 0$, $Q_{[f]}(w,w) = 0$, $\partial_1 H(w,w) = 0$.*

Then, there exists a unique $u \in L^\infty(Q_T)$ (which is said to be the weak entropy solution of problem (3.1)–(3.3)) satisfying, for all "boundary entropy-flux pair" $(H, Q_{[f]})$:

$$\left(\mathcal{P}_1^{(a)}\right) \begin{cases} \displaystyle\int_{Q_T} H(u,k)\ \frac{\partial \varphi}{\partial t} + Q_{[f]}(u,k)\ \nabla\varphi\ dx\ dt + \int_\Omega H(u^0,k)\ \varphi(0,x)\ dx \\ \qquad\qquad\qquad\qquad\qquad\qquad \displaystyle +\mathcal{L}_{[f]} \int_{\Sigma_T} H(u^D,k)\ \varphi\ d\gamma(r)\ dt \geq 0, \\ \forall \phi \in \mathcal{D}((-\infty,T)\times\mathbb{R}^d),\ \phi \geq 0,\ k \in \mathbb{R}. \end{cases}$$

Moreover, u is a function with values in

$$\left[\min\left(\inf_\Omega u^0, \inf_{\Sigma_T} u^D\right), \max\left(\sup_\Omega u^0, \sup_{\Sigma_T} u^D\right)\right].$$

Actually, the class of "boundary entropy-flux pairs" can be replaced by the class of "semi Kružkov entropy-flux pairs" [Ser96, Vov02]:

$$\Big((u-k)^{\pm}, \operatorname{sgn}_{\pm}(u-k)(f(u)-f(k))\Big)$$

which gives rise to the same notion of "weak entropy solution" as defined by Otto (as explained further):

Definition-Theorem 3 (Vovelle [Vov02]). *Suppose that*

(i") $f \in \left(C^2(\mathbb{R})\right)^d$ (i.e. the flux is autonomous, only depends on the variable u), $g \equiv 0$,

(ii") f is Lipschitz continuous,

(iii") $(u^0, u^D) \in L^\infty(\Omega; [a,b]) \times L^\infty(\Sigma_T; [a,b])$.

Then, there exists a unique $u \in L^\infty(Q_T)$ (which is said to be the weak entropy solution of problem (3.1)–(3.3)) satisfying:

$$\left(\mathcal{P}_1^{(b)}\right)\begin{cases} \displaystyle\int_{Q_T} (u-k)^{\pm}\ \frac{\partial \varphi}{\partial t} + \big(\operatorname{sgn}_{\pm}(u-k)\ (f(u)-f(k))\big)\ \nabla\varphi\ dx\ dt \\ \displaystyle\qquad\qquad + \int_\Omega (u^0-k)^{\pm}\ \varphi(0,x)\ dx \\ \displaystyle\qquad\qquad + \mathcal{L}_{[f]} \int_{\Sigma_T} (u^D-k)^{\pm}\ \varphi\ d\gamma(r)\ dt \geq 0, \\ \forall \phi \in \mathcal{D}((-\infty,T)\times\mathbb{R}^d),\ \phi \geq 0,\ \forall k \in \mathbb{R}. \end{cases}$$

Moreover, u is a function with values in

$$\left[\min\left(\inf_\Omega u^0, \inf_{\Sigma_T} u^D\right), \max\left(\sup_\Omega u^0, \sup_{\Sigma_T} u^D\right)\right].$$

As previously mentioned, in the definition of a weak entropy solution, Otto used "boundary entropy-flux pairs" instead of "semi Kružkov entropy-flux pairs". However, thanks to the following lemma, Definition-Theorem 3 of weak entropy solution used by Vovelle (i.e. with "semi Kružkov entropy-flux pairs"), is equivalent to Definition-Theorem 2 used by Otto (i.e. with "boundary entropy-flux pairs").

Lemma 3.1.

(i) Let $\eta \in C^1(\mathbb{R};\mathbb{R})$ be a convex function such that there exists $w \in [a,b]$ with $\eta(w) = 0$ and $\eta'(w) = 0$. Then η can be uniformly approximated on $[a,b]$ by applications of the kind

$$s \longmapsto \sum_1^p \alpha_i (s-\kappa_i)^- + \sum_1^q \beta_j (s-\tilde{\kappa}_j)^+$$

where $\alpha_i \geq 0$, $\beta_j \geq 0$, $\kappa_i \in [a,b]$ *and* $\tilde{\kappa}_j \in [a,b]$.

(ii) Conversely, there exists a sequence of "boundary entropy-flux pairs" $\{(H_\delta^\pm, Q_{[f],\delta}^\pm)\}_\delta$ *(see further (3.11) and (3.13)), and letting* $\delta \to 0$*, such that the following uniform convergence holds:* $(H_\delta^\pm(z,k), Q_{[f],\delta}^\pm(z,k)) \longrightarrow ((z-k)^\pm, \Phi_{[f]}^\pm(z,k))$.

Notice that replacing "semi Kružkov entropy-flux pairs" by "Kružkov entropy-flux pairs" in the formulation of $(\mathcal{P}_1^{(b)})$ would not be sufficient: indeed, the class of "Kružkov entropy-flux pairs" is not wide enough to ensure uniqueness of the weak entropy solution (see the counter-example of [Vov02]). Thus, it appears that "semi Kružkov entropy-flux pairs" / "boundary entropy-flux pairs" play a crucial role to ensure existence and uniqueness of the weak entropy solution.

This work is organized as follows:

3.1 Definition, initial and boundary conditions, set preserving property,

3.2 Existence,

3.3 Uniqueness.

Existence and uniqueness theorems are based on techniques that have been widely used in [Kru70, BLRN79, Ott96, MNRR96]. But we point out the fact that these arguments have never been gathered with the appropriate definition of a weak entropy solution in this general framework in order to establish an existence and uniqueness theorem along with a set preserving result: in fact, we deeply use the results detailed in [MNRR96], up to the following modifications: proofs for existence and uniqueness are adapted to the "semi Kružkov entropy-flux pairs", dealing with additional terms induced by the source term g and the fact that the flux f is non-autonomous.

3.1 Definition, initial / boundary conditions, set preserving property

Definition 5. *Let us suppose that Assumption 13 holds. A function $u \in L^\infty(Q_T, [a,b])$ is said to be a weak entropy solution of problem (3.1)–(3.3) if it satisfies*

$$(\mathcal{P}_{SK}) \begin{cases} \displaystyle\int_{Q_T} \Big\{ (u-k)^\pm \frac{\partial \varphi}{\partial t} & + \quad \big(\mathrm{sgn}_\pm(u-k)(f(t,x,u) - f(t,x,k))\big) \ \nabla\varphi \\ & - \quad \mathrm{sgn}_\pm(u-k)\Big(\nabla \cdot f(t,x,k) + g(t,x,u)\Big) \ \varphi \Big\} \ dx \ dt \\ & + \quad \displaystyle\int_\Omega (u^0 - k)^\pm \ \varphi(0,x) \ dx \\ & + \quad \mathcal{L}_{[f]} \displaystyle\int_{\Sigma_T} (u^D - k)^\pm \ \varphi \ d\gamma(r) \ dt \geq 0 \\ \forall \phi \in \mathcal{D}((-\infty, T) \times \mathbb{R}^d), \ \phi \geq 0, \ \forall k \in \mathbb{R}. \end{cases}$$

This definition leads to the following remark:

Remark 3.2.

- *The additive terms due to the non-homogeneous property of the flux and the source term are obviously similar to the ones which appear in the work of Kružkov [Kru70] and Bardos, Le Roux and Nédélec [BLRN79]. Indeed, up to the boundary term, the main difference concerns the entropy-flux pairs: the "Kružkov entropy-flux pairs" (which appear in [Kru70, BLRN79]) have turned into "semi Kružkov entropy-flux pairs".*

- *A function u which satisfies Definition 5 is a weak solution in a classical sense. Indeed, for every $\varphi \in H_0^1(Q_T)$, we write $\varphi = \varphi^+ - \varphi^-$, with $\varphi^+ = \max(\varphi, 0)$ and $\varphi^- = -\min(\varphi, 0)$; obviously, $\varphi^\pm \in H_0^1(Q_T)$; thus adding the two inequalities (corresponding to each "semi Kružkov entropy-flux pair") gives:*

$$\int_{Q_T} \Big\{ \Big|u-k\Big| \frac{\partial \varphi^\pm}{\partial t} + \mathrm{sgn}(u-k)\Big(f(t,x,u) - f(t,x,k)\Big) \ \nabla\varphi^\pm \\ -\mathrm{sgn}(u-k) \ \Big(\nabla \cdot f(t,x,k) + g(t,x,u)\Big) \ \varphi^\pm \Big\} \ dx \ dt \geq 0.$$

Taking $k = \pm\|u\|_{L^\infty(Q_T)}$ gives

$$\int_{Q_T} \Big\{ u \ \frac{\partial \varphi^\pm}{\partial t} + f(t,x,u) \ \nabla\varphi^\pm - g(t,x,u) \ \varphi^\pm \Big\} \ dx \ dt = 0.$$

Now, by means of substraction, we immediatly obtain

$$\int_{Q_T} \left\{ u \; \frac{\partial \varphi}{\partial t} + f(t,x,u) \; \nabla\varphi - g(t,x,u) \; \varphi \right\} dx \; dt = 0.$$

for every $\varphi \in H_0^1(Q_T)$, so that Equation (3.1) is gained in a weak sense.

- *When f is an autonomous flux and $g \equiv 0$, passing from the "semi Kružkov entropy-flux pairs" formulation (Definition-Theorem 3) to the "boundary entropy-flux pairs" formulation (Definition-Theorem 2) - and conversely - is straigthforward by Lemma 3.1 and linearity. In the non-autonomous case, the additive term*

$$-\int_{Q_T} \operatorname{sgn}(u-k)^{\pm} \Big(\nabla \cdot f(t,x,k) + g(t,x,u) \Big) \; \varphi \; dx \; dt$$

prevents us from establishing a "boundary entropy-flux pairs" formulation. Thus, it is not so clear that the methods used in Otto's work may be easily adapted to the non-autonomous case.

Now, let us explain the way the boundary / initial conditions are satisfied. Interestingly, the concept of "boundary entropy-flux pairs" defined by Otto is still the key point. Thus let us slightly generalize the definition of "boundary entropy-flux pairs":

Definition 6. *Let $(H, Q_{[f]})$ be in $C^1(\mathbb{R}^2) \times \left(C^1((0,T) \times \overline{\Omega} \times \mathbb{R}^2)\right)^d$. The pair $(H, Q_{[f]})$ is said to be a "boundary entropy-flux pair" (for the flux f) if:*

1. *for all $w \in \mathbb{R}$, $s \mapsto H(s,w)$ is a convex function,*
2. *$\forall w \in \mathbb{R}$, $\partial_s Q_{[f]}(\cdot,\cdot,s,w) = \partial_s H(s,w) \dfrac{\partial f}{\partial s}(\cdot,\cdot,s)$,*
3. *$\forall w \in \mathbb{R}$, $H(w,w) = 0$, $Q_{[f]}(\cdot,\cdot,w,w) = 0$, $\partial_s H(w,w) = 0$.*

Lemma 3.3 (Boundary condition)**.** *Let $u \in L^\infty(Q_T)$ satisfying $(\mathcal{P}_{SK})$. Then,*

$$\operatorname*{ess\,lim}_{\varrho \to 0^+} \int_{\Sigma_T} Q_{[f]}(t,r,u(t,r-\varrho\; n(r)), u^D(t,r)) \cdot n(r) \; \beta(t,r) \; d\gamma(r) \; dt \geq 0, \tag{3.7}$$

for all "boundary entropy-flux pair" $(H, Q_{[f]})$, $\forall \beta \in L^1(\Sigma_T)$, $\beta \geq 0$ a.e.

Proof. We directly use the proof of Lemma 7.12 in [MNRR96], adapted to the particular case of the "semi Kružkov entropy-flux pairs". Thus, we easily state that if $u \in L^\infty(Q_T)$ satisfies $(\mathcal{P}_{SK})$, then, defining the quantity

$$\operatorname*{ess\,lim}_{\varrho \to 0^+} \int_{\Sigma_T} \Big\{ \operatorname{sgn}_{\pm}(u(t,r-\varrho\; n(r)) - v^D(t,r)) \Big(f(t,r,u(t,r-\varrho\; n(r)))$$

$$-f(t,r,v^D(t,r)))\Big)\Bigg\} \cdot n(r)\ \beta(t,r)\ d\gamma(r)\ dt \tag{3.8}$$

exists for all $\beta \in L^1((0,T)\times\mathbb{R}^{d-1})$, $\beta \geq 0$ a. e., and all $v^D \in L^\infty((0,T)\times\mathbb{R}^{d-1})$. Moreover, we have:

$$\begin{aligned}\operatorname*{ess\,lim}_{\varrho\to 0^+} \int_{\Sigma_T} \Bigg\{ &\mathrm{sgn}_\pm(u(t,r-\varrho\, n(r)) - v^D(t,r))\Big(f(t,r,u(t,r-\varrho\, n(r))) \\ &-f(t,r,v^D(t,r)))\Big)\Bigg\} \cdot n(r)\ \beta(t,r)\ d\gamma(r)\ dt \\ &\geq -\mathcal{L}_{[f]} \int_{\Sigma_T} (u^D(t,r) - v^D(t,r))^\pm\ \beta(t,r)\ d\gamma(r)\ dt,\end{aligned}$$

for all $\beta \in L^1((0,T)\times\mathbb{R}^{d-1})$, $\beta \geq 0$ a. e., and all $v^D \in L^\infty((0,T)\times\mathbb{R}^{d-1})$. Then, taking $v^D = u^D$, every "boundary flux" $Q_{[f]}$ is uniformly approached by a linear combination of "semi Kružkov fluxes" (see Lemma 3.1), every coefficient being non-negative, which preserves the inequality and concludes the proof. ☐

To complete the scope of boundary / initial conditions, we recall the following result, which is proved by using the same arguments than in Lemma 7.41 of [MNRR96]:

Lemma 3.4 (Initial condition). *Let $u \in L^\infty(Q_T)$ satisfying $(\mathcal{P}_{SK})$. Then,*

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_\Omega \Big|u(t,x) - u^0(x)\Big|\ dx = 0 \tag{3.9}$$

Now let us give some comprehensive details on the way the boundary condition is satisfied:

Remark 3.5. *The boundary condition 3.7 is nothing less than the one obtained in [Ott96, MNRR96], up to a generalization to non-autonomous fluxes and taking account of a source-term which does not interfere in the boundary condition. We have proved that it is satisfied, although working only with the "semi Kružkov entropy-flux pairs" formulation (let us recall that a "boundary entropy-flux pairs" formulation is not possible anymore). However the way to understand the boundary condition is given in [MNRR96, Ott96, Vov02]: generally speaking, the problem should be overdetermined and the boundary equality cannot be required to be assumed at each point of the boundary, even if the solution is a regular function. But, with additional assumptions, the more comprehensive "BLN" condition is recovered:*

(i) *If u admits a trace, meaning that there exists $u_{|\Sigma_T} \in L^\infty(\Sigma_T)$ such that*

$$\operatorname{ess}\lim_{\varrho\to 0^+} \int_{\Sigma_T} |u(\tau, r - \varrho\, n(r)) - u_{|\Sigma_T}(\tau, r)|\, d\gamma(r)\, d\tau = 0$$

then Equation (3.7) is equivalent to the following equation (see [DL88, Ott96])

$$Q_{[f]}(\cdot,\cdot,u_{|\Sigma_T},u^D)\cdot n \geq 0, \quad \text{a.e. on } \Sigma_T.$$

Considering the particular boundary fluxes

$$H_\delta^+(z,\kappa) = \Big((\max(z-\kappa,0))^2 + \delta^2\Big)^{1/2} - \delta, \tag{3.10}$$

$$Q_{[f],\delta}^+(\cdot,\cdot,z,\kappa) = \int_\kappa^z \partial_1 H_\delta^+(\lambda,k)\, \frac{\partial f}{\partial u}(\cdot,\cdot,\lambda)\, d\lambda \tag{3.11}$$

and

$$H_\delta^-(z,\kappa) = \Big((\min(\kappa-z,0))^2 + \delta^2\Big)^{1/2} - \delta, \tag{3.12}$$

$$Q_{[f],\delta}^-(\cdot,\cdot,z,\kappa) = \int_\kappa^z \partial_1 H_\delta^-(\lambda,k)\, \frac{\partial f}{\partial u}(\cdot,\cdot,\lambda)\, d\lambda, \tag{3.13}$$

letting $\delta \to 0$, we obtain the following uniform convergences:

$$Q_{[f],\delta}^\pm(\cdot,\cdot,z,\kappa) \to \operatorname{sgn}_\pm(z-\kappa)(f(\cdot,\cdot,z) - f(\cdot,\cdot,\kappa)).$$

Finally taking the boundary flux

$$\begin{aligned} Q_{[f]}(\cdot,\cdot,s,w) \quad = \quad & \operatorname{sgn}_+(s - \max(w,k))(f(\cdot,\cdot,s) - f(\cdot,\cdot,\max(w,k))) \\ & + \operatorname{sgn}_-(s - \min(w,k))(f(\cdot,\cdot,s) - f(\cdot,\cdot,\min(w,k))) \end{aligned}$$

yields the classical condition given by Bardos, Le Roux and Nédélec [BLRN79], that is:

$$\begin{aligned} &\textit{for a.e. } (t,r) \in \Sigma_T,\ \forall k \in [\min(u_{|\Sigma_T}, u^D), \max(u_{|\Sigma_T}, u^D)], \\ &\operatorname{sgn}(u_{|\Sigma_T}(t,r) - u^D(t,r))(f(t,r,u_{|\Sigma_T}(t,r)) - f(t,r,k))\cdot n(r) \geq 0. \end{aligned} \tag{3.14}$$

(ii) *Assume furthermore that for almost every $(t,r) \in \Sigma_T$, $s \mapsto f(t,r,s)\cdot n(r)$ is a monotone function. Then, Inequality (3.14) can be simplified in specific cases: indeed, one has*

$$u = u^D, \quad \text{on } \Sigma_T^D = \left\{(t,r) \in \Sigma_T,\ \frac{\partial f}{\partial u}(t,r,u)\cdot n(r) < 0\right\}$$

and nothing is imposed at $\Sigma_T^N = \left\{(t,r) \in \Sigma_T,\ \dfrac{\partial f}{\partial u}(t,r,u)\cdot n(r) \geq 0\right\}$.

Thus the boundary condition is "active" only on a part of the boundary.

Before stating the existence / uniqueness result in next sections, we prove the following property:

Theorem 3.6 (Set preserving property)**.** *Under Assumption 13 (we recall that, in particular,* u^0*,* u^D *are functions with values in* $[a,b]$*), if a function* u *satisfies* $(\mathcal{P}_{SK})$*, then* $a \leq u \leq b$ *a.e. on* Q_T*.*

Proof. Set $k = a$ in $(\mathcal{P}_{SK})$. Since we have by Assumption 13 (iii) and (iv),

$$(u^0 - a)^- = 0, \quad (u^D - a)^- = 0,$$

the boundary / initial terms vanish. Then if we choose a particular test-function which only depends on time t, we obtain:

$$\int_{Q_T} \left\{(u-a)^-\phi'(t) - \operatorname{sgn}_-(u-a)\Big(\nabla\cdot f(t,x,a) + g(t,x,u)\Big)\ \phi(t)\right\}\ dx\ dt \geq 0,$$

for all $\phi \in \mathcal{D}([0,T[)$, $\phi \geq 0$. Now, using

$$\Big(\nabla\cdot f(t,x,a) + g(t,x,u)\Big) = \Big(\nabla\cdot f(t,x,a) + g(t,x,a)\Big) + g(t,x,u) - g(t,x,a)$$

and Assumption 13 (iv), we get

$$\int_{Q_T} (u-a)^-\ \phi'(t) - \operatorname{sgn}_-(u-a)\left(g(t,x,u) - g(t,x,a)\right)\ \phi(t)\ dx\ dt \geq 0, \tag{3.15}$$

for all $\phi \in \mathcal{D}([0,T[)$, $\phi \geq 0$. Furthermore, it can be easily proved that the following property holds:

$$-\mathcal{L}_{[g]}(u-a)^- \leq \operatorname{sgn}_-(u-a)\left(g(t,x,u) - g(t,x,a)\right) \leq \mathcal{L}_{[g]}(u-a)^-$$

and Inequality (3.15) implies

$$\int_{Q_T} (u-a)^-\left(\phi'(t) + \mathcal{L}_{[g]}\phi(t)\right)\ dx\ dt \geq 0.$$

Introducing the function

$$q_a(t) = e^{-\mathcal{L}_{[g]}t}\int_\Omega (u-a)^-(t,x)\ dx, \tag{3.16}$$

the earlier inequality gives

$$\int_0^T q_a(t) e^{\mathcal{L}_{[g]} t} \left(\phi'(t) + \mathcal{L}_{[g]}\phi(t)\right) \, dt \geq 0,$$

Denoting $\psi(t) = e^{\mathcal{L}_{[g]} t}\phi(t)$, we infer that

$$\int_0^T q_a(t) \; \psi'(t) \; dx \; dt \geq 0, \tag{3.17}$$

for all $\psi \in \mathcal{D}([0, T[)$, $\psi \geq 0$. Let $\tau < T$, $\delta_\tau = T - \tau$ and $r \in \mathcal{D}([0, T[)$ be such that: r is non-increasing, $r \equiv 1$ on $[0, \tau]$, $r \equiv 0$ on $[\tau + \delta_\tau/2, T[$. Choosing

$$\psi(t) = \frac{T-t}{T} \, r(t)$$

in Inequality (3.17) gives

$$-\frac{1}{T}\int_0^T q_a(t) \; r(t) \; dt + \int_0^T q_a(t) \; \frac{T-t}{T} \; r'(t) \; dt \geq 0.$$

Since $r' \leq 0$, the second term of the left-hand side of the previous inequality is negative. Since $r(t) = 1$, $\forall t \in (0, \tau)$ and $r \geq 0$, the first term is upper bounded by

$$-\frac{1}{T}\int_0^\tau q_a(t) \; dt$$

which is consequently non-negative. But, q_a is obviously a non-negative function, so that

$$q_a \equiv 0, \text{on } (0, \tau).$$

Therefore, we deduce from the definition of q_a (see Equation (3.16)) that $(u-a)^- = 0$ on $\Omega \times (0, \tau)$. Letting $\tau \to T$, we have $u \geq a$ a.e. Similarly, by choosing $k = b$ in $(\mathcal{P}_{SK})$ (with the "semi Kružkov entropy" $u \mapsto (u-b)^+$), we prove $u \leq b$ a.e. □

Remark 3.7. *Under Assumption 13 (iv), only the restriction to the set $[a, b]$ of functions $s \mapsto f(t, x, s)$ and $s \mapsto g(t, x, s)$ plays a role in the behaviour of a weak entropy solution. Therefore, it would still make sense to weaken Assumption 13 (i), by considering functions $f(t, x, \cdot)$ and $g(t, x, \cdot)$ only defined on $[a, b]$ instead of $\mathbb{R}$.*

In next sections, we prove that under Assumption 13, problem (3.1)–(3.3) admits a unique weak entropy solution (in the sense of Definition 5). Existence is proved using the vanishing viscosity method (parabolic approximation) (see Section 3.2) and uniqueness is stated with Kružkov's double variable method (see Section 3.3).

3.2 Existence

Existence is obtained from a classical parabolic approximation (vanishing viscosity method). We consider the following set of equations:

$$\frac{\partial u_\varepsilon}{\partial t} + \nabla \cdot \Big(f(t,x,u_\varepsilon)\Big) + g(t,x,u_\varepsilon) = \varepsilon \Delta u_\varepsilon, \quad \text{on } Q_T, \tag{3.18}$$

$$u_\varepsilon(0,\cdot) = u_\varepsilon^0, \quad \text{on } \Omega, \tag{3.19}$$

$$u = u_\varepsilon^D, \quad \text{on } \Sigma_T, \tag{3.20}$$

where the following assumption holds:

Assumption 14.

(i) u_ε^D *and* u_ε^0 *satisfy compatibility conditions on* $\overline{\Sigma}_T \cap \overline{Q}_T$*,*

(ii) u_ε^D *and* u_ε^0 *are smooth functions: at least,* $u_\varepsilon^D \in C^2(\Sigma_T;[a,b])$*,* $u_\varepsilon^0 \in C^2(\overline{\Omega};[a,b])$*.*

Under Assumption 14, the quasilinear parabolic problem (3.18)–(3.20) admits a unique solution $u_\varepsilon \in C^2(\overline{\Omega}\times]0,T[)$. We study the convergence of $\{u_\varepsilon\}_\varepsilon$ when ε tends to 0. As in [MNRR96], we introduce the following tools:

Definition 7. *Let μ be a sufficient small positive number, and let us define the following function:*

$$s(x) = \begin{cases} \min\left(\text{dist}(x,\partial\Omega),\mu\right), & \text{if } x \in \Omega, \\ -\min\left(\text{dist}(x,\partial\Omega),\mu\right), & \text{if } x \in \mathbb{R}^d \setminus \Omega. \end{cases}$$

Let ξ_ε be defined by

$$\xi_\varepsilon(x) = 1 - \exp\left(-\frac{\mathcal{L}_{[f]} + \varepsilon\mathcal{R}}{\varepsilon}\ s(x)\right), \quad \text{with} \quad \mathcal{R} = \sup_{0<s(x)<\mu} |\Delta s(x)|.$$

Notice that s is Lipschitz continuous in $\mathbb{R}^d$ and smooth on the closure of the set

$$\{x \in \mathbb{R}^d,\ |s(x)| < \mu\}.$$

Moreover, it can be proved (see [MNRR96]) that the following property holds:

Proposition 3.8. *Let ξ_ε be defined in Definition 7. Then, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\varphi \geq 0$,*

$$\mathcal{L}_{[f]} \int_\Omega \Big|\nabla \xi_\varepsilon\Big|\ \varphi \leq \varepsilon \int_\Omega \nabla \xi_\varepsilon \nabla \varphi + (\mathcal{L}_{[f]} + \varepsilon\mathcal{R}) \int_{\partial\Omega} \varphi, \tag{3.21}$$

Lemma 3.9. *Let (u, u^D, u^0) satisfy equations (3.18)–(3.20), the boundary / initial conditions satisfying Assumption 14 (subscripts are dropped for convenience). Then,*

(i) for all $\varphi \in \mathcal{D}(]-\infty, T[\times\mathbb{R}^d)$, for all $k \in \mathbb{R}$,

$$
\begin{aligned}
&\int_{Q_T} \Big\{ (u-k)^\pm \frac{\partial \varphi}{\partial t} + \mathrm{sgn}_\pm(u-k)\Big(f(t,x,u) - f(t,x,k)\Big)\ \nabla\varphi \\
&\quad -\mathrm{sgn}_\pm(u-k)\Big(\nabla\cdot f(t,x,k) + g(t,x,u)\Big)\varphi + \varepsilon\ (u-k)^\pm \Delta\varphi \Big\}\ \xi_\varepsilon\ dx\ dt \\
&\quad + \int_\Omega (u^0 - k)^\pm\ \varphi(0,x)\ \xi_\varepsilon\ dx \\
&\quad \geq -2\varepsilon \int_{Q_T} (u-k)^\pm\ \nabla\varphi\ \nabla\xi_\varepsilon\ dx\ dt - (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_{\Sigma_T} (u^D - k)^\pm\ \varphi\ d\gamma(r)\ dt,
\end{aligned}
\tag{3.22}
$$

(ii) the following set preserving property holds:

$$
a \leq u \leq b. \tag{3.23}
$$

Proof. ■ *Proof of (i):* Let us define the functions:

$$
\begin{aligned}
\mathrm{sgn}^\eta_\pm(z) &= \begin{cases} H_\eta(z), & \text{if } z \in \mathbb{R}^\pm, \\ -H_\eta(-z), & \text{if } z \in \mathbb{R}^\mp, \end{cases} \\
I^\pm_\eta(z) &= \int_0^z \mathrm{sgn}^\eta_\pm(t)\ dt,
\end{aligned}
$$

where the function H_η is a classical approximation of the Heaviside graph:

$$
H_\eta(z) = \begin{cases} 0, & \text{if } z \leq 0, \\ z/\eta, & \text{if } 0 < z \leq \eta, \\ 1, & \text{if } z > \eta. \end{cases}
$$

Obviously, the pairs

$$
\Big(I^\pm_\eta(z,k), \mathrm{sgn}^\eta_\pm(z-k)\Big(f(t,x,z) - f(t,x,k)\Big)\Big)
$$

mimic the behaviour of the "semi Kružkov entropy-flux pairs"

$$
\Big((z-k)^\pm, \mathrm{sgn}_\pm(z-k)\Big(f(t,x,z) - f(t,x,k)\Big)\,.
$$

Notice that $I^\pm_\eta(\cdot,k) \in C^1(\mathbb{R})$ is piecewise convex. Then, multiplying Equation (3.18) by

$\mathrm{sgn}^\eta_\pm(u-k)\ \varphi\ \xi_\varepsilon$, with $\varphi \in \mathcal{D}(]-\infty,T[\times\mathbb{R}^d)$, we obtain (after integration by parts):

$$\begin{aligned}
\int_{Q_T} \Big\{ I^\pm_\eta(u,k)\frac{\partial\varphi}{\partial t} &+ \mathrm{sgn}^\eta_\pm(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\nabla\varphi \\
&- \ \mathrm{sgn}^\eta_\pm(u-k)\Big(\nabla\cdot f(t,x,k)+g(t,x,u)\Big)\varphi + \varepsilon\ I^\pm_\eta(u,k)\Delta\varphi\Big\}\ \xi_\varepsilon \\
&+ \ \int_{Q_T} \mathrm{sgn}^\eta_\pm(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\ \varphi\ \nabla\xi_\varepsilon \\
&+ \ \int_\Omega \Big(\int_k^{u^0} \mathrm{sgn}^\eta_\pm(v-k)\ dv\Big)\ \varphi(0,\cdot)\ \xi_\varepsilon \\
&+ \ \int_{Q_T} \Big(f(t,x,u)-f(t,x,k)\Big)\cdot\nabla u\ {\mathrm{sgn}^\eta_\pm}'(u-k)\ \varphi\ \xi_\varepsilon \\
&\geq \ \varepsilon\int_{Q_T} \Big\{\nabla\left(I^\pm_\eta(u,k)\ \varphi\right)\nabla\xi_\varepsilon - 2I^\pm_\eta(u,k)\nabla\varphi\nabla\xi_\varepsilon\Big\}.
\end{aligned}$$

After some computation, we state that:

$$\Big|\mathrm{sgn}^\eta_\pm(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\Big| \leq \mathcal{L}_{[f]}\ I^\pm_\eta(u,k) + \mathcal{L}_{[f]}\ \eta,$$

so that the inequality becomes

$$\begin{aligned}
\int_{Q_T} \Big\{ I^\pm_\eta(u,k)\ \frac{\partial\varphi}{\partial t} &+ \mathrm{sgn}^\eta_\pm(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\ \nabla\varphi \\
&- \ \mathrm{sgn}^\eta_\pm(u-k)\Big(\nabla\cdot f(t,x,k)+g(t,x,u)\Big)\ \varphi + \varepsilon\ I^\pm_\eta(u,k)\ \Delta\varphi\Big\}\ \xi_\varepsilon \\
&+ \ \int_\Omega \Big(\int_k^{u^0} \mathrm{sgn}^\eta_\pm(v-k)\ dv\Big)\ \varphi(0,\cdot)\ \xi_\varepsilon \\
&+ \ \int_{Q_T} \Big(f(t,x,u)-f(t,x,k)\Big)\cdot\nabla u\ {\mathrm{sgn}^\eta_\pm}'(u-k)\ \varphi\ \xi_\varepsilon \\
&\geq \ \varepsilon\int_{Q_T} \Big\{\nabla\left(I^\pm_\eta(u,k)\ \varphi\right)\ \nabla\xi_\varepsilon - 2I^\pm_\eta(u,k)\ \nabla\varphi\ \nabla\xi_\varepsilon\Big\} \\
&- \ \mathcal{L}_{[f]}\ \int_{Q_T} I^\pm_\eta(u,k)\ \varphi\ \Big|\nabla\xi_\varepsilon\Big| - \mathcal{L}_{[f]}\ \eta\int_{Q_T}\varphi\ \Big|\nabla\xi_\varepsilon\Big|.
\end{aligned}$$

Using Proposition 3.8 with $I^\pm_\eta(u,k)\varphi$ instead of φ, we get:

$$\begin{aligned}
\mathcal{L}_{[f]}\ \int_{Q_T} I^\pm_\eta(u,k)\ \varphi\ \Big|\ \nabla\xi_\varepsilon\Big| \\
\leq \varepsilon\int_{Q_T}\nabla\left(I^\pm_\eta(u,k)\ \varphi\right)\ \nabla\xi_\varepsilon + (\mathcal{L}_{[f]}+\mathcal{R}\varepsilon)\int_{\Sigma_T} I^\pm_\eta(u^D,k)\ \varphi.
\end{aligned}$$

Using this result in the previous inequality gives:

$$\begin{aligned}
\int_{Q_T} \Big\{ & I_\eta^\pm(u,k)\ \frac{\partial\varphi}{\partial t} + \mathrm{sgn}_\pm^\eta(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\ \nabla\varphi \\
& - \ \mathrm{sgn}_\pm^\eta(u-k)\Big(\nabla\cdot f(t,x,k)+g(t,x,u)\Big)\ \varphi + \varepsilon\ I_\eta^\pm(u,k)\ \Delta\varphi \Big\}\ \xi_\varepsilon \\
& + \ \int_\Omega \Big(\int_k^{u^0} \mathrm{sgn}_\pm^\eta(v-k)\ dv\Big)\ \varphi(0,\cdot)\ \xi_\varepsilon \\
& + \ \int_{Q_T} \Big(f(t,x,u)-f(t,x,k)\Big)\cdot\nabla u\ \mathrm{sgn}_\pm^{\eta\,\prime}(u-k)\ \varphi\ \xi_\varepsilon \\
& \geq \ -2\varepsilon \int_{Q_T} I_\eta^\pm(u,k)\ \nabla\varphi\ \nabla\xi_\varepsilon - (\mathcal{L}_{[f]}+\mathcal{R}\varepsilon)\int_{\Sigma_T} I_\eta^\pm(u^D,k)\ \varphi \\
& \quad -\mathcal{L}_{[f]}\ \eta \int_{Q_T} \varphi\ \Big|\nabla\xi_\varepsilon\Big|.
\end{aligned}$$

Now, let η tend to 0. The first and second terms of the left-hand side give:

$$\begin{aligned}
\int_{Q_T} \Big\{ & (u-k)^\pm\ \frac{\partial\varphi}{\partial t} + \mathrm{sgn}_\pm(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\ \nabla\varphi \\
& - \ \mathrm{sgn}_\pm(u-k)\Big(\nabla\cdot f(t,x,k)+g(t,x,u)\Big)\ \varphi + \varepsilon\ (u-k)^\pm\ \Delta\varphi \Big\}\ \xi_\varepsilon \\
& + \ \int_\Omega (u^0-k)^\pm \varphi(0,\cdot)\ \xi_\varepsilon.
\end{aligned}$$

The last term of the left-hand side tends to 0 by Lemma 2 in [BLRN79][3]. Finally, the right-hand side term tends to

$$-2\varepsilon \int_{Q_T} (u-k)^\pm\ \nabla\varphi\ \nabla\xi_\varepsilon - (\mathcal{L}_{[f]}+\mathcal{R}\varepsilon)\int_{\Sigma_T} (u^D-k)^\pm\ \varphi$$

and the proof is concluded.

■ *Proof of* (ii)*:* Taking into account the properties of f and g (see Assumption 13) and $u_\varepsilon^D, u_\varepsilon^0$ (see Assumption 14), we choose in the first inequality stated in this lemma (see (i)) the particular value of k, namely $k=a$, with a test function which only depends on time t. Thus, we obtain

$$\int_{Q_T} (u_\varepsilon - a)^- \varphi'(t)\ dx\ dt \geq 0,$$

[3]This lemma, due to Saks [Sak64], says that if $v \in C^1(\overline{\Omega})$, then

$$\lim_{\eta\to 0}\int_\Omega |\nabla v|\ \chi_\eta = 0$$

χ_η being the characteristic function of the set $\{x\in\Omega;\ |v(x)|\leq\eta\}$.

for all $\varphi \in \mathcal{D}(]-\infty, T[)$, $\varphi \geq 0$. Then, similarly to the proof of Theorem 3.6, we obtain that $u_\varepsilon \geq a$ a.e. In the same way, we prove that $u_\varepsilon \leq b$ a.e. $\square$

Now we state the following L^1-stability result:

Lemma 3.10. *Let (u_1, u_1^D, u_1^0), (u_2, u_2^D, u_2^0), satisfy equations (3.18)–(3.20), the corresponding boundary / initial conditions satisfying Assumption 14. Then,*

$$\int_\Omega \Big|u_1(t,\cdot) - u_2(t,\cdot)\Big| \ \xi_\varepsilon \leq \left\{ \int_\Omega \Big|u_1^0 - u_2^0\Big| \ \xi_\varepsilon + (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_{\Sigma_T} \Big|u_1^D - u_2^D\Big| \right\} e^{\mathcal{L}_{[g]}T}, \tag{3.24}$$

for all $t \in (0,T)$.

Proof. Let us denote $w = u_1 - u_2$, $w^D = u_1^D - u_2^D$, $w^0 = u_1^0 - u_2^0$. Multiplying the equation

$$\frac{\partial w}{\partial t} + \nabla \cdot \Big(f(t,x,u_1) - f(t,x,u_2)\Big) + \Big(g(t,x,u_1) - g(t,x,u_2)\Big) - \varepsilon \Delta w = 0$$

by $\varphi_\delta{}'(w)\xi_\varepsilon$, where $\varphi_\delta(z) = (z^2 + \delta^2)^{1/2}$, and integrating over $(0,t) \times \Omega$, we obtain

$$\begin{aligned}
\int_\Omega \varphi_\delta(w(t,\cdot)) \ \xi_\varepsilon - \int_\Omega \varphi_\delta(w^0) \ \xi_\varepsilon & \\
- \int_0^t \int_\Omega \Big(f(\tau,x,u_1) - f(\tau,x,u_2)\Big)\Big(\varphi_\delta{}''(w) \ \nabla w \ \xi_\varepsilon + \varphi_\delta{}'(w) \ \nabla \xi_\varepsilon\Big) & \\
+ \int_0^t \int_\Omega \Big(g(\tau,x,u_1) - g(\tau,x,u_2)\Big) \ \varphi_\delta{}'(w) \ \xi_\varepsilon & \\
+ \ \varepsilon \int_0^t \int_\Omega \Big\{ \Big|\nabla w\Big|^2 \varphi_\delta{}''(w) \ \xi_\varepsilon + (\nabla \varphi_\delta(w)) \ \nabla \xi_\varepsilon \Big\} &= 0.
\end{aligned}$$

Now we study the behaviour of each term w.r.t δ: using the uniform Lipschitz continuity of f, Young's inequality and the fact that $z^2\varphi_\delta{}''(z) = z^2\delta^2(z^2+\delta^2)^{-3/2} < \delta$, we get

$$\begin{aligned}
&-\Big(f(\tau,x,u_1) - f(\tau,x,u_2)\Big) \ \nabla w \ \varphi_\delta{}''(w) \ \xi_\varepsilon + \varepsilon \Big|\nabla w\Big|^2 \varphi_\delta{}''(w) \ \xi_\varepsilon \\
&\geq \Big\{ -\mathcal{L}_{[f]}|w| \ |\nabla w| + \varepsilon \Big|\nabla w\Big|^2 \Big\} \varphi_\delta{}''(w) \ \xi_\varepsilon \geq -\frac{\mathcal{L}_{[f]}^2}{4\varepsilon} w^2 \varphi_\delta{}''(w) \ \xi_\varepsilon \geq -\frac{\mathcal{L}_{[f]}^2 \ \delta}{4\varepsilon} \xi_\varepsilon.
\end{aligned}$$

Moreover, observing that $|z|\varphi_\delta{}'(z) \leq \varphi_\delta(z)$, we obtain

$$\begin{aligned}
-\Big(f(\tau,x,u_1) - f(\tau,x,u_2)\Big) \ \varphi_\delta{}'(w) \ \nabla \xi_\varepsilon &\geq -\mathcal{L}_{[f]} \ |w| \ |\varphi_\delta{}'(w)| \ |\nabla \xi_\varepsilon| \\
&\geq -\mathcal{L}_{[f]} \ \varphi_\delta(w) |\nabla \xi_\varepsilon|.
\end{aligned}$$

Following the same idea, we get

$$\Big(g(\tau,x,u_1) - g(\tau,x,u_2)\Big) \ \varphi_\delta{}'(w) \ \xi_\varepsilon \geq -\mathcal{L}_{[g]} \ |w| \ |\varphi_\delta{}'(w)| \ \xi_\varepsilon \geq -\mathcal{L}_{[g]} \ \varphi_\delta(w) \ \xi_\varepsilon.$$

Finally, using the previous inequalities, we state that

$$\begin{aligned}\int_\Omega \varphi_\delta(w(t,\cdot))\ \xi_\varepsilon - \int_\Omega \varphi_\delta(w^0)\ \xi_\varepsilon \\ - \frac{\mathcal{L}_{[f]}^2 \delta\ T}{4\varepsilon} \int_\Omega \xi_\varepsilon - \mathcal{L}_{[f]} \int_0^t \int_\Omega \varphi_\delta(w)\ |\nabla \xi_\varepsilon| \\ - \mathcal{L}_{[g]} \int_0^t \int_\Omega \varphi_\delta(w)\ \xi_\varepsilon + \varepsilon \int_0^t \int_\Omega \nabla\left(\varphi_\delta(w)\right)\ \nabla\xi_\varepsilon \leq 0\end{aligned}$$

and, therefore, taking $\varphi = \varphi_\delta(w)$ in Inequality (3.21) gives

$$\begin{aligned}\int_\Omega \varphi_\delta(w(t,\cdot))\ \xi_\varepsilon \ \leq\ & \int_\Omega \varphi_\delta(w^0)\ \xi_\varepsilon + \mathcal{L}_{[g]} \int_0^t \int_\Omega \varphi_\delta(w)\ \xi_\varepsilon \\ & + (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_0^t \int_\Omega \varphi_\delta(w^D) + \frac{\mathcal{L}_{[f]}^2 \delta\ T}{4\varepsilon} \int_\Omega \xi_\varepsilon.\end{aligned}$$

Now let δ tend to 0. We get

$$\begin{aligned}\int_\Omega |w(t,\cdot)|\ \xi_\varepsilon \ &\leq\ \int_\Omega |w^0|\ \xi_\varepsilon + (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_0^t \int_\Omega |w^D| + \mathcal{L}_{[g]} \int_0^t \int_\Omega |w|\ \xi_\varepsilon \\ &\leq\ \int_\Omega |w^0|\ \xi_\varepsilon + (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_{\Sigma_T} |w^D| + \mathcal{L}_{[g]} \int_0^t \int_\Omega |w|\ \xi_\varepsilon.\end{aligned}$$

Applying Gronwall's lemma concludes the proof. □

Lemma 3.11. *Let (u, u^D, u^0) satisfy equations (3.18)–(3.20), the corresponding boundary / initial conditions satisfying Assumption 14. We suppose furthermore that u^D has a smooth extension to $\overline{Q}_T$, denoted $\overline{u}^D$. Then, there exists a constant λ which only depends on $\|u^0\|_\Omega$, $\|\overline{u}^D\|_{\Sigma_T}$, T, Ω, f and g such that*

$$\sup_{t\in(0,T)} \int_\Omega \left\{ \left|\frac{\partial u}{\partial t}(t,\cdot)\right| + \left|\nabla u(t,\cdot)\right| \right\} \leq \lambda, \tag{3.25}$$

where we have used the notation

$$\begin{aligned}\left\|u^0\right\|_\Omega = & \int_\Omega \left|\Delta u^0\right| + \left|\nabla u^0\right| + \left|u^0\right|, \\ \left\|\overline{u}^D\right\|_{\Sigma_T} = & \sup_{Q_T} \left\{ \left|\Delta \overline{u}^D\right| + \left|\frac{\partial \overline{u}^D}{\partial t}\right| + \left|\nabla \overline{u}^D\right| + \left|\overline{u}^D\right| \right\} \\ & + \int_{Q_T} \left|\nabla^2 \frac{\partial \overline{u}^D}{\partial t}\right| + \left|\nabla^3 \overline{u}^D\right| + \left|\frac{\partial^2 \overline{u}^D}{\partial t^2}\right| + \left|\nabla \frac{\partial \overline{u}^D}{\partial t}\right| + \left|\nabla^2 \overline{u}^D\right|.\end{aligned}$$

Proof. In this proof, we will say that a constant "does not depend on ε" if it only depends on $\|u^0\|_\Omega$, $\|\overline{u}^D\|_{\Sigma_T}$, T, Ω, f and g. Moreover, for the sake of simplicity, $\overline{u}^D$ will be identified to u^D. The proof is organized in two steps:

- Step 1: Boundness of $\displaystyle\int_\Omega \Big|\frac{\partial u}{\partial t}(t,\cdot)\Big|$,
- Step 2: Boundness of $\displaystyle\int_\Omega \Big|\nabla u(t,\cdot)\Big|$.

■ **Step 1: Boundness of** $\displaystyle\int_\Omega \Big|\frac{\partial u}{\partial t}(t,\cdot)\Big|$

Let us still denote u^D the smooth extension of u^D onto $\overline{Q}_T$. We introduce

$$v = u - u^D$$

$$\begin{aligned} e \; = \; & \frac{\partial^2 u^D}{\partial t^2} \; + \; \nabla\cdot\left(\frac{\partial f}{\partial u}(\cdot,\cdot,u)\frac{\partial u^D}{\partial t}\right) + \nabla\cdot\left(\frac{\partial f}{\partial t}(\cdot,\cdot,u)\right) \\ & + \; \frac{\partial g}{\partial u}(\cdot,\cdot,u)\frac{\partial u^D}{\partial t} + \frac{\partial g}{\partial t}(\cdot,\cdot,u) - \varepsilon\Delta\frac{\partial u^D}{\partial t}, \end{aligned}$$

so that we easily get

$$\frac{\partial^2 v}{\partial t^2} + \nabla\cdot\left(\frac{\partial f}{\partial u}(\cdot,\cdot,u)\frac{\partial v}{\partial t}\right) + \frac{\partial g}{\partial u}(\cdot,\cdot,u)\;\frac{\partial v}{\partial t} - \varepsilon\Delta\left(\frac{\partial v}{\partial t}\right) = -e. \tag{3.26}$$

Multiplying Equation (3.26) by

$$\varphi_\delta{}'\left(\frac{\partial v}{\partial t}\right),$$

where $\varphi_\delta(z) = (z^2+\delta^2)^{1/2}$, and integrating over $(0,t)\times\Omega$, we obtain

$$\begin{aligned} & \int_\Omega \varphi_\delta\left(\frac{\partial v}{\partial t}(t,\cdot)\right) - \int_\Omega \varphi_\delta\left(\frac{\partial v}{\partial t}(0,\cdot)\right) - \int_0^t\int_\Omega \frac{\partial f}{\partial u}(\tau,x,u)\cdot\nabla\left(\frac{\partial v}{\partial t}\right)\;\frac{\partial v}{\partial t}\;\varphi_\delta{}''\left(\frac{\partial v}{\partial t}\right) \\ & + \int_0^t\int_\Omega \frac{\partial g}{\partial u}(\tau,x,u)\;\frac{\partial v}{\partial t}\;\varphi_\delta{}'\left(\frac{\partial v}{\partial t}\right) + \varepsilon\int_0^t\int_\Omega \Big|\nabla\left(\frac{\partial v}{\partial t}\right)\Big|^2\varphi_\delta{}''\left(\frac{\partial v}{\partial t}\right) \\ & = -\int_0^t\int_\Omega e\;\varphi_\delta{}'\left(\frac{\partial v}{\partial t}\right), \end{aligned} \tag{3.27}$$

by using the property $\varphi_\delta{}'\,(\partial v/\partial t) = 0$ on Σ_T. Further, we have

$$\begin{aligned} & -\frac{\partial f}{\partial u}(\tau,x,u)\cdot\nabla\left(\frac{\partial v}{\partial t}\right)\;\frac{\partial v}{\partial t}\;\varphi_\delta{}''\left(\frac{\partial v}{\partial t}\right) + \varepsilon\Big|\nabla\left(\frac{\partial v}{\partial t}\right)\Big|^2\varphi_\delta{}''\left(\frac{\partial v}{\partial t}\right) \\ & \qquad \geq -\frac{1}{4\varepsilon}\Big|\frac{\partial f}{\partial u}(\tau,x,u)\Big|^2\;\left(\frac{\partial v}{\partial t}\right)^2\;\varphi_\delta{}''\left(\frac{\partial v}{\partial t}\right) \geq -\frac{\mathcal{L}^2_{[f]}\;\delta}{4\varepsilon}. \end{aligned}$$

Thus, letting $\delta \to 0$ in Equation (3.27) gives

$$\int_\Omega \Big|\frac{\partial v}{\partial t}(t,\cdot)\Big| \le \int_\Omega \Big|\frac{\partial v}{\partial t}(0,\cdot)\Big| + \int_0^t \int_\Omega |e| + \mathcal{L}_{[g]} \int_0^t \int_\Omega \Big|\frac{\partial v}{\partial t}\Big|$$

which obviously implies

$$\int_\Omega \Big|\frac{\partial u}{\partial t}(t,\cdot)\Big| \le \int_\Omega \Big|\frac{\partial u^D}{\partial t}(t,\cdot)\Big| + \int_\Omega \Big|\frac{\partial v}{\partial t}(0,\cdot)\Big| + \int_0^t \int_\Omega |e| + \mathcal{L}_{[g]} \int_0^t \int_\Omega \Big|\frac{\partial v}{\partial t}\Big|. \tag{3.28}$$

Now, let us briefly analyse each term of the right-hand side in the previous inequality:

- ▶ **(step 1)** *Analysis of* $\int_\Omega \Big|\frac{\partial u^D}{\partial t}(t,\cdot)\Big|$ - It is obviously bounded by $c_1 = \|u^D\|_{\Sigma_T}$.

- ▶ **(step 1)** *Analysis of* $\int_\Omega \Big|\frac{\partial v}{\partial t}(0,\cdot)\Big|$ - We obtain from equation (3.18)

$$\int_\Omega \Big|\frac{\partial v}{\partial t}(0,\cdot)\Big| \quad = \quad \int_\Omega \Big| -\nabla\cdot\Big(f(0,\cdot,u^0)\Big) - g(0,\cdot,u^0) + \varepsilon\Delta u^0 - \frac{\partial u^D}{\partial t}(0,\cdot)\Big|.$$

So far, we have

$$\begin{aligned}\int_\Omega \Big| -\nabla\cdot\Big(f(0,\cdot,u^0)\Big)\Big| \quad &= \quad \int_\Omega \Big\{ \Big|\left(\nabla\cdot f\right)(0,\cdot,u^0)\Big)\Big| \quad + \quad \Big|\frac{\partial f}{\partial u}(0,\cdot,u^0)\ \nabla u^0\Big| \Big\} \\ &\le \quad \mathcal{L}_{[\nabla\cdot f]} \int_\Omega |u^0| \quad + \quad \mathcal{L}_{[f]} \int_\Omega |\nabla u^0| \\ &\le \quad c_2^{(1)}\end{aligned}$$

where $c_2^{(1)}$ only depends on f and $\|u^0\|_\Omega$. Moreover,

$$\int_\Omega \Big|g(0,\cdot,u^0)\Big| \le |\Omega|\ \sup\Big(\Big|g(t,x,s)\Big|,\ (t,x,s) \in \overline{Q}_T \times [a,b]\Big) \le c_2^{(2)},$$

where $c_2^{(2)}$ only depends on g and Ω. Further, for ε bounded (which can be assumed, for instance $\varepsilon \le 1$), we get

$$\int_\Omega \Big|\varepsilon\Delta u^0 - \frac{\partial u^D}{\partial t}(0,\cdot)\Big| \le \int_\Omega \Big|\Delta u^0\Big| + |\Omega|\ \sup_{Q_T} \Big|\frac{\partial u^D}{\partial t}\Big| \le c_2^{(3)},$$

where $c_2^{(3)}$ only depends on $\|u^0\|_\Omega$, $\|u^D\|_{\Sigma_T}$ and Ω. Thus, the analysed term satisfies:

$$\int_\Omega \Big|\frac{\partial v}{\partial t}(0,\cdot)\Big| \le c_2,$$

where c_2 only depends on f, g, $\|u^0\|_\Omega$, $\|u^D\|_{\Sigma_T}$ and Ω.

▶ **(step 1)** *Analysis of* $\displaystyle\int_0^t \int_\Omega |e|$ - Let us recall that, from the definition of e:

$$\int_0^t \int_\Omega |e| \leq \int_0^t \int_\Omega \Bigg\{ \Big|\frac{\partial^2 u^D}{\partial t^2}\Big| + \Big|\nabla \cdot \Big(\frac{\partial f}{\partial u}(\cdot,\cdot,u)\frac{\partial u^D}{\partial t}\Big)\Big| + \Big|\nabla \cdot \Big(\frac{\partial f}{\partial t}(\cdot,\cdot,u)\Big)\Big| + \Big|\frac{\partial g}{\partial u}(\cdot,\cdot,u)\frac{\partial u^D}{\partial t}\Big| + \Big|\frac{\partial g}{\partial t}(\cdot,\cdot,u)\Big| + \Big|\varepsilon\Delta\frac{\partial u^D}{\partial t}\Big| \Bigg\}.$$

Now, we have

$$\int_0^t \int_\Omega \Big|\frac{\partial^2 u^D}{\partial t^2}\Big| \leq \int_{Q_T} \Big|\frac{\partial^2 u^D}{\partial t^2}\Big| \leq c_3^{(1)}$$

with, for instance, $c_3^{(1)} = \|u^D\|_{\Sigma_T}$. Moreover, one has

$$\nabla \cdot \Big(\frac{\partial f}{\partial u}(\cdot,\cdot,u)\frac{\partial u^D}{\partial t}\Big) = \Big(\nabla \cdot \frac{\partial f}{\partial u}\Big)(\cdot,\cdot,u)\frac{\partial u^D}{\partial t} + \frac{\partial^2 f}{\partial u^2}(\cdot,\cdot,u)\cdot\nabla u\,\frac{\partial u^D}{\partial t} + \frac{\partial f}{\partial u}(\cdot,\cdot,u)\cdot\nabla\frac{\partial u^D}{\partial t}.$$

Thus, each term can be controlled in the following way:

$$\begin{aligned}
\int_0^t \int_\Omega \Big|\Big(\nabla \cdot \frac{\partial f}{\partial u}\Big)(\cdot,\cdot,u)\frac{\partial u^D}{\partial t}\Big| &\leq c_3^{(2)},\\
\int_0^t \int_\Omega \Big|\frac{\partial^2 f}{\partial u^2}(\cdot,\cdot,u)\cdot\nabla u\,\frac{\partial u^D}{\partial t}\Big| &\leq c_3^{(3)} \int_0^t \int_\Omega \Big|\nabla u\Big|,\\
\int_0^t \int_\Omega \Big|\frac{\partial f}{\partial u}(\cdot,\cdot,u)\cdot\nabla\frac{\partial u^D}{\partial t}\Big| &\leq c_3^{(4)},
\end{aligned}$$

with, for instance,

$$\begin{aligned}
c_3^{(2)} &= \sup_{Q_T\times[a,b]}\Big|\nabla\cdot\frac{\partial f}{\partial u}\Big| \sup_{Q_T}\Big|\frac{\partial u^D}{\partial t}\Big|\ T\ |\Omega|,\\
c_3^{(3)} &= \sup_{Q_T\times[a,b]}\Big|\frac{\partial^2 f}{\partial u^2}\Big| \sup_{Q_T}\Big|\frac{\partial u^D}{\partial t}\Big|,\\
c_3^{(4)} &= \mathcal{L}_{[f]}\int_{Q_T}\Big|\nabla\frac{\partial u^D}{\partial t}\Big|.
\end{aligned}$$

Further again,

$$\begin{aligned}
\int_0^t \int_\Omega \Big|\nabla \cdot \Big(\frac{\partial f}{\partial t}(\cdot,\cdot,u)\Big)\Big| &\leq \int_0^t \int_\Omega \Big|\nabla\cdot\frac{\partial f}{\partial t}(\cdot,\cdot,u)\Big| + \Big|\frac{\partial^2 f}{\partial t\,\partial u}(\cdot,\cdot,u)\cdot\nabla u\Big|\\
&\leq c_3^{(5)} + c_3^{(6)}\int_\Omega\int_0^t\Big|\nabla u\Big|
\end{aligned}$$

with, for instance,

$$\begin{aligned} c_3^{(5)} &= \sup_{Q_T\times[a,b]} \Big|\nabla\cdot\frac{\partial f}{\partial t}\Big|\ T\ |\Omega|, \\ c_3^{(6)} &= \sup_{Q_T\times[a,b]} \Big|\frac{\partial^2 f}{\partial u^2}\Big|\ \sup_{Q_T}\Big|\frac{\partial u^D}{\partial t}\Big|. \end{aligned}$$

Finally, we have also

$$\int_0^t\int_\Omega \Big|\frac{\partial g}{\partial u}(\cdot,\cdot,u)\frac{\partial u^D}{\partial t}\Big| + \Big|\frac{\partial g}{\partial t}(\cdot,\cdot,u)\Big| + \Big|\varepsilon\Delta\frac{\partial u^D}{\partial t}\Big| \le c_3^{(7)},$$

with

$$c_3^{(7)} = \left(\mathcal{L}_{[g]}\sup_{Q_T}\Big|\frac{\partial u^D}{\partial t}\Big| + \sup_{Q_T\times[a,b]}\Big|\frac{\partial g}{\partial t}\Big| + \sup_{Q_T}\Big|\Delta\frac{\partial u^D}{\partial t}\Big|\right)\ T\ |\Omega|.$$

Taking

$$c_3 = \max\left(c_3^{(1)} + c_3^{(2)} + c_3^{(4)} + c_3^{(5)} + c_3^{(7)}, c_3^{(3)} + c_3^{(6)}\right)$$

and using the previous inequalities together gives, we get

$$\int_0^t\int_\Omega |e| \le c_3\left(1 + \int_\Omega\int_0^t \Big|\nabla u\Big|\right),$$

c_3 being a constant only depending on f, g, Ω, T and $\|u^D\|_{\Sigma_T}$.

▶ **(step 1)** *Analysis of* $\mathcal{L}_{[g]}\displaystyle\int_0^t\int_\Omega\Big|\frac{\partial v}{\partial t}\Big|$ - We have, obviously, the following property:

$$\begin{aligned} \mathcal{L}_{[g]}\int_0^t\int_\Omega\Big|\frac{\partial v}{\partial t}\Big| &\le \mathcal{L}_{[g]}\left(\int_0^t\int_\Omega\Big|\frac{\partial u}{\partial t}\Big| + \int_0^t\int_\Omega\Big|\frac{\partial u^D}{\partial t}\Big|\right) \\ &\le c_4\left(1 + \int_0^t\int_\Omega\Big|\frac{\partial u}{\partial t}\Big|\right), \end{aligned}$$

with

$$c_4 = \mathcal{L}_{[g]}\max\left(1, \sup_{Q_T}\Big|\frac{\partial u^D}{\partial t}\Big|\ |\Omega|\ T\right).$$

Thus, recalling Inequality (3.28) along with the previous results, we obtain

$$\int_\Omega\Big|\frac{\partial u}{\partial t}(t,\cdot)\Big| \le c_5\left(1 + \int_0^t\int_\Omega|\nabla u| + \int_0^t\int_\Omega\Big|\frac{\partial u}{\partial t}\Big|\right) \tag{3.29}$$

by taking, for instance, $c_5 = \displaystyle\sum_{i=1}^4 c_i$ which does not depend on ε.

■ **Step 2: Boundness of** $\displaystyle\int_\Omega\Big|\nabla u(t,\cdot)\Big|$

For this, we proceed in two steps, namely Steps $2^{(a)}$ and $2^{(b)}$, which will be gathered in order to conclude **Step 2**.

Let us proceed to **Step** $2^{(a)}$. Recalling that $v = u - u^D$ and denoting

$$h_1 = \frac{\partial u^D}{\partial t} + \nabla \cdot (f(t,x,u^D)) + g(t,x,u^D) - \varepsilon \Delta u^D, \tag{3.30}$$

we have

$$\frac{\partial v}{\partial t} + \nabla \cdot (f(t,x,u) - f(t,x,u^D)) + g(t,x,u) - g(t,x,u^D) - \varepsilon \Delta v = -h_1. \tag{3.31}$$

We multiply Equation (3.31) by ${\varphi_\delta}'(v)\ \beta$, where $\beta \in \mathcal{D}(\mathbb{R})$, $\beta \geq 0$, depends only on the space variable and $\varphi_\delta(z) = (z^2 + \delta^2)^{1/2} - \delta$.. After integration over $(0,t) \times \Omega$, partial integration and using the fact that ${\varphi_\delta}'(v) = 0$, $\varphi_\delta(v) = 0$, $\nabla \varphi_\delta(v) \cdot n = 0$ on Σ_T, we obtain

$$\begin{aligned}
\int_\Omega \varphi_\delta(v(t,\cdot))\ \beta - \int_\Omega \varphi_\delta(v(0,\cdot))\ \beta - \varepsilon \int_0^t \int_\Omega \varphi_\delta(v)\ \Delta\beta& \\
- \int_0^t \int_\Omega {\varphi_\delta}'(v)\ (f(\tau,x,u) - f(\tau,x,u^D))\nabla\beta& \\
- \int_0^t \int_\Omega (f(\tau,x,u) - f(\tau,x,u^D))\nabla v\ {\varphi_\delta}''(v)\ \beta& \\
+ \int_0^t \int_\Omega {\varphi_\delta}'(v)\ (g(\tau,x,u) - g(\tau,x,u^D))\beta& \\
+\varepsilon \int_0^t \int_\Omega |\nabla v|^2\ {\varphi_\delta}''(v)\ \beta = - \int_0^t \int_\Omega {\varphi_\delta}'(v)\ h_1\ \beta&.
\end{aligned}$$

We let $\delta \to 0$ and thus

$$\begin{aligned}
\int_\Omega |v(t,\cdot)|\ \beta - \int_\Omega |v(0,\cdot)|\ \beta - \varepsilon \int_0^t \int_\Omega |v|\ \Delta\beta& \\
- \int_0^t \int_\Omega \operatorname{sgn}(u - u^D)\ (f(\tau,x,u) - f(\tau,x,u^D))\nabla\beta& \\
+ \int_0^t \int_\Omega \operatorname{sgn}(u - u^D)\ (g(\tau,x,u) - g(\tau,x,u^D))\beta& \\
\leq - \int_0^t \int_\Omega \operatorname{sgn}(u - u^D)\ h_1\ \beta&.
\end{aligned} \tag{3.32}$$

Now we choose

$$\beta(x) = \gamma\left(\frac{s(x)}{\rho}\right),$$

where $s(x)$ is defined as before, ρ is a strictly positive number and $\gamma \in \mathcal{D}(\mathbb{R})$ is a fixed non-negative function such that

$$\gamma(0) = 0, \quad \gamma(\sigma) = 1, \text{ for } \sigma \geq 1.$$

Let us study the behaviour with respect to ρ of each term:

► **(step $2^{(a)}$)** *Behaviour with respect to ρ of* $\int_0^t \int_\Omega \operatorname{sgn}(u-u^D)\ (f(t,x,u)-f(t,x,u^D))\nabla\beta$*:*

Obviously, one has

$$\nabla\beta = \gamma'\left(\frac{s(x)}{\rho}\right)\ \frac{1}{\rho}\nabla s(x)$$

and

$$\nabla s(x) = 0, \quad \text{on } \Omega \setminus K_\mu$$

with $K_\mu = \{x \in \Omega,\ \operatorname{dist}(x, \partial\Omega) < \mu\}$. Thus, each point $x \in K_\mu$ (for μ small enough) can be described as $x = r(x) - s(x)\ n(r)$, where $r(x)$ is the nearest boundary point to x, and $n(r)$ is the outer vector to $\partial\Omega$ at point $r(x)$. Let us notice that $\nabla s(x) = -n(r)$, if $x \in K_\mu$. From the previous observations, we deduce the following equatlity (for the sake of simplicity, $F(u, u^D)(\tau, x)$ denotes the value of the function

$$\operatorname{sgn}(u - u^D)\ (f(\cdot,\cdot,u) - f(\cdot,\cdot,u^D))$$

at point $(\tau, x) \in Q_T$):

$$\begin{aligned}
&\int_0^t \int_\Omega F(u,u^D)(\tau,x)\ \nabla\beta(x)\ dx\ d\tau \\
&= \int_0^t \int_{K_\mu} F(u,u^D)(\tau,x)\ \gamma'\left(\frac{s(x)}{\rho}\right)\ \frac{1}{\rho}\nabla s(x)\ dx\ d\tau \\
&= \int_0^t \int_0^\mu \int_{\partial\Omega} F(u,u^D)(\tau, r - sn(r))\ \gamma'\left(\frac{s}{\rho}\right)\ \frac{1}{\rho}\ (-n(r))\ d\gamma(r)\ ds\ d\tau \\
&= -\int_0^t \int_0^{\mu/\rho} \int_{\partial\Omega} F(u,u^D)(\tau, r - \sigma\rho\ n(r))\ \gamma'(\sigma)\ n(r)\ d\gamma(r)\ d\sigma\ d\tau \\
&= -\int_0^{\mu/\rho} \gamma'(\sigma)\left(\int_0^t \int_{\partial\Omega} F(u,u^D)(\tau, r - \sigma\rho\ n(r))\ n(r)\ d\gamma(r)\ d\tau\right)\ d\sigma.
\end{aligned}$$

Thus, letting $\rho \to 0$, we obtain

$$\begin{aligned}
&\lim_{\rho\to 0}\int_0^t\int_\Omega F(u,u^D)(\tau,x)\ \nabla\beta(x)\ dx\ d\tau\\
&\quad= -\int_0^{+\infty}\gamma'(\sigma)\ \left(\int_0^t\int_{\partial\Omega}F(u,u^D)(\tau,r)\ n(r)\ d\gamma(r)\ d\tau\right)d\sigma\\
&\quad= -\int_0^{+\infty}\gamma'(\sigma)\ d\sigma\ \left(\int_0^t\int_{\partial\Omega}F(u,u^D)(\tau,r)\ n(r)\ d\gamma(r)\ d\tau\right)\\
&\quad= 0,
\end{aligned}$$

since $F(u,u^D)=0$ on Σ_T.

► **(step $2^{(a)}$)** *Behaviour with respect to ρ of $\displaystyle\int_0^t\int_\Omega |v|\ \Delta\beta$:*

For the particular choice of β, we have:

$$\begin{aligned}
\Delta\beta &= \nabla\cdot\left(\frac{1}{\rho}\gamma'\left(\frac{s(x)}{\rho}\right)\ \nabla s(x)\right)\\
&= \frac{1}{\rho}\sum_{i=1}^d\left\{\frac{1}{\rho}\gamma''\left(\frac{s(x)}{\rho}\right)\left(\frac{\partial s(x)}{\partial x_i}\right)^2+\gamma'\left(\frac{s(x)}{\rho}\right)\ \frac{\partial^2 s(x)}{\partial x_i^2}\right\}.
\end{aligned}$$

Moreover, if $x\in K_\mu$, then $\dfrac{\partial s(x)}{\partial x_i}=-n_i(r)$, n_i being the ith component of $n(r)$, so that

$$\frac{1}{\rho^2}\gamma''\left(\frac{s(x)}{\rho}\right)\sum_{i=1}^d\left(\frac{\partial s(x)}{\partial x_i}\right)^2=\frac{1}{\rho^2}\gamma''\left(\frac{s(x)}{\rho}\right)\|n(r)\|^2=\frac{1}{\rho^2}\gamma''\left(\frac{s(x)}{\rho}\right)$$

and, as a consequence,

$$\Delta\beta=\begin{cases}\dfrac{1}{\rho^2}\gamma''\left(\dfrac{s(x)}{\rho}\right)+\dfrac{1}{\rho}\gamma'\left(\dfrac{s(x)}{\rho}\right)\ \Delta s(x), & \text{on } K_\mu,\\ 0, & \text{elsewhere.}\end{cases}$$

Thus, since $v(\tau,r(x))=0$ ($r(x)$ being a boundary point) and using the previous expression of $\Delta\beta$, we have

$$\begin{aligned}
\int_0^t\int_\Omega\Big|v\Big|\ \Delta\beta &= \int_0^t\int_\Omega\Big|v(\tau,x)-v(\tau,r(x))\Big|\ \Delta\beta\\
&= \int_0^t\int_{K_\mu}\Big|v(\tau,x)-v(\tau,r(x))\Big|\left(\gamma''\left(\frac{s(x)}{\rho}\right)\frac{1}{\rho^2}+\frac{1}{\rho}\gamma'\left(\frac{s(x)}{\rho}\right)\Delta s(x)\right).
\end{aligned}$$

Let us focus on the first right-hand side term of the previous equality:

$$\begin{aligned}
&\int_0^t \int_0^\mu \int_{\partial\Omega} \Big| v(\tau, r - s\, n(r)) - v(\tau, r) \Big| \, \gamma'' \left(\frac{s}{\rho}\right) \frac{1}{\rho^2} \, d\gamma(r) \, ds \, d\tau \\
&= \int_0^t \int_0^{\mu/\rho} \int_{\partial\Omega} \frac{\Big| v(\tau, r - \sigma\rho\, n(r)) - v(\tau, r) \Big|}{\rho} \, \gamma''(\sigma) \, d\gamma(r) \, d\sigma \, d\tau \\
&= \int_0^{\mu/\rho} \sigma \, \gamma''(\sigma) \, \left(\int_0^t \int_{\partial\Omega} \frac{\Big| v(\tau, r - \sigma\rho\, n(r)) - v(\tau, r) \Big|}{\sigma\rho} \, d\gamma(r) \, d\tau \right) d\sigma.
\end{aligned}$$

Now let us focus on the second right-hand side: since $|\Delta s| \leq \mathcal{R}$ on K_μ,

$$\begin{aligned}
&\Big| \int_0^t \int_{K_\mu} \Big| v(\tau, x) - v(\tau, r(x)) \Big| \, \frac{1}{\rho} \, \gamma' \left(\frac{s(x)}{\rho}\right) \, \Delta s(x) \, dx \, d\tau \, \Big| \\
&\leq \mathcal{R} \int_0^t \int_0^\mu \int_{\partial\Omega} \Big| v(\tau, r - s\, n(r)) - v(\tau, r) \Big| \, \frac{1}{\rho} \, \Big| \gamma' \left(\frac{s}{\rho}\right) \Big| \, d\gamma(r) \, ds \, d\tau \\
&\leq \mathcal{R} \int_0^t \int_0^{\mu/\rho} \int_{\partial\Omega} \Big| v(\tau, r - \sigma\rho\, n(r)) - v(\tau, r) \Big| \, \gamma'(\sigma) \, d\gamma(r) \, d\sigma \, d\tau \\
&= \mathcal{R} \, \rho \int_0^{\mu/\rho} \sigma \, \gamma'(\sigma) \, \left(\int_0^t \int_{\partial\Omega} \frac{\Big| v(\tau, r - \sigma\rho\, n(r)) - v(\tau, r) \Big|}{\sigma\rho} \, d\gamma(r) \, d\tau \right) d\sigma.
\end{aligned}$$

Letting $\rho \to 0$ (notice that the second right-hand side term tends to 0) gives

$$\begin{aligned}
\lim_{\rho \to 0} \int_0^t \int_\Omega \Big| v \Big| \, \Delta\beta &= \int_0^{+\infty} \sigma \, \gamma''(\sigma) \, \left(\int_0^t \int_{\partial\Omega} \Big| \nabla v(\tau, r) \cdot n(r) \Big| \, d\gamma(r) \, d\tau \right) d\sigma \\
&= -\int_0^t \int_{\partial\Omega} \Big| \nabla v(\tau, r) \cdot n(r) \Big| \, d\gamma(r) \, d\tau
\end{aligned}$$

As a consequence, Inequality (3.32) becomes

$$\int_\Omega |v(t, \cdot)| + \varepsilon \int_{\partial\Omega} \Big| \nabla v \cdot n \Big| \leq \int_\Omega |v(0, \cdot)| + \mathcal{L}_{[g]} \int_0^t \int_\Omega |v| + \int_0^t \int_\Omega |h_1|. \tag{3.33}$$

Now, we proceed to **Step** $2^{(b)}$. Let us denote

$$z_i = \frac{\partial v}{\partial x_i}, \quad z = \nabla v.$$

Then we have

$$\frac{\partial z_i}{\partial t} + \nabla \cdot \left(\frac{\partial f}{\partial u}(t, x, u) \, z_i \right) - \varepsilon \Delta z_i = -h_2^{(i)},$$

with

$$h_2^{(i)} = \frac{\partial^2 u^D}{\partial x_i \partial t} + \nabla\cdot\left(\frac{\partial f}{\partial u}(t,x,u)\ \frac{\partial u^D}{\partial x_i}\right) \quad + \quad \nabla\cdot\left(\frac{\partial f}{\partial x_i}(t,x,u)\right)$$
$$+ \quad \frac{\partial}{\partial x_i}\Big(g(t,x,u)\Big) - \varepsilon\Delta\frac{\partial u^D}{\partial x_i}.$$

Multiplying the previous equation by $\partial\phi_\delta/\partial\xi_i(z)$, with $\phi_\delta(\xi) = (|\xi|^2+\delta^2)^{1/2}$, adding the terms $(i = 1, d)$, we have, using the usual Einstein summation convention (i.e. whenever an index appears twice in one expression, the summation over this index is performed):

$$\int_0^t\int_\Omega \frac{\partial z_i}{\partial t}\frac{\partial\phi_\delta}{\partial\xi_i}(z) = \int_\Omega \phi_\delta(v(t,\cdot)) - \int_\Omega \phi_\delta(v(0,\cdot)),$$
$$\int_0^t\int_\Omega \Delta z_i\frac{\partial\phi_\delta}{\partial\xi_i}(z) = -\int_0^t\int_\Omega \frac{\partial z_i}{\partial x_j}\frac{\partial^2\phi_\delta}{\partial\xi_i\partial\xi_k}(z)\frac{\partial z_k}{\partial x_j} + \int_0^t\int_{\partial\Omega}\frac{\partial z_i}{\partial x_j}n_j\frac{\partial\phi_\delta}{\partial\xi_i}(z),$$
$$\int_0^t\int_\Omega \nabla\cdot\left(\frac{\partial f}{\partial u}(\cdot,\cdot,u)\ z_i\right)\frac{\partial\phi_\delta}{\partial\xi_i}(z)$$
$$= -\int_0^t\int_\Omega \frac{\partial f_j}{\partial u}(\cdot,\cdot,u)\ z_i\ \frac{\partial^2\phi_\delta}{\partial\xi_i\partial\xi_k}(z)\frac{\partial z_k}{\partial x_j} + \int_0^t\int_{\partial\Omega}\frac{\partial f_j}{\partial u}(\cdot,\cdot,u)n_j\ z_i\ \frac{\partial\phi_\delta}{\partial\xi_i}(z).$$

Due to the estimate

$$\varepsilon\frac{\partial z_i}{\partial x_j}\frac{\partial^2\phi_\delta}{\partial\xi_i\partial\xi_k}(z)\frac{\partial z_k}{\partial x_j} - \frac{\partial f_j}{\partial u}(\cdot,\cdot,u)\ z_i\ \frac{\partial^2\phi_\delta}{\partial\xi_i\partial\xi_k}(z)\frac{\partial z_k}{\partial x_j}$$
$$= \frac{\delta^2}{(|z|^2+\delta^2)^{3/2}}\left[\varepsilon|\nabla z|^2 - \frac{\partial f_j}{\partial u}(\cdot,\cdot,u)\ z_i\ \frac{\partial z_i}{\partial x_j}\right]$$
$$\geq -\frac{1}{4\varepsilon}\left|\frac{\partial f}{\partial u}(\cdot,\cdot,u)\right|^2\frac{\delta^2|z|^2}{(|z|^2+\delta^2)^{3/2}} \geq -\frac{\mathcal{L}_{[f]}\ \delta}{4\varepsilon},$$

we obtain for $\delta\to 0$

$$\int_\Omega\Big|z(t,\cdot)\Big| \ \leq\ \int_\Omega\Big|z(0,\cdot)\Big| + \int_0^t\int_\Omega|h_2|$$
$$+\limsup_{\delta\to 0}\int_0^t\int_{\partial\Omega}\left|\frac{\partial f}{\partial u}(\cdot,\cdot,u)\cdot n\ z_j\ \frac{\partial\phi_\delta}{\partial\xi_j}(z) - \varepsilon\nabla z_i\cdot n\ \frac{\partial\phi_\delta}{\partial\xi_i}(z)\right|.$$

Due to $z = 0$ on Σ_T, we have on Σ_T

$$z = \nabla v = (\nabla v\cdot n)\ n, \quad \Delta v = D^2v(n,n) + \Delta s\nabla v\cdot n,$$

where D^2v is the bilinear form of the second differential of v. Therefore, the integrand can be rewritten as

$$\begin{aligned}
&\frac{\partial f}{\partial u}(\cdot,\cdot,u)\cdot n\ z_j\ \frac{\partial \phi_\delta}{\partial \xi_j}(z) - \varepsilon \nabla z_i \cdot n\ \frac{\partial \phi_\delta}{\partial \xi_i}(z)\\
&\qquad = \frac{\partial f}{\partial u}(\cdot,\cdot,u)\cdot n\ \frac{|\nabla v|^2}{(|\nabla v|^2+\delta^2)^{1/2}} - \varepsilon D^2 v\left(n, \frac{\nabla v}{(|\nabla v|^2+\delta^2)^{1/2}}\right)\\
&\qquad = \left(\frac{\partial f}{\partial u}(\cdot,\cdot,u)\cdot \nabla v - \varepsilon D^2 v(n,n)\right)\frac{\nabla v\cdot n}{(|\nabla v|^2+\delta^2)^{1/2}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\nabla\cdot\big(f(t,x,u)-f(t,x,u^D)\big)\\
&\qquad = (\nabla\cdot f)(t,x,u) - (\nabla\cdot f)(t,x,u^D) + \frac{\partial f}{\partial u}\cdot\nabla u - \frac{\partial f}{\partial u}\cdot \nabla u^D\\
&\qquad = (\nabla\cdot f)(t,x,u) - (\nabla\cdot f)(t,x,u^D) + \frac{\partial f}{\partial u}(t,x,u)\cdot\nabla v\\
&\qquad\qquad - \left(\frac{\partial f}{\partial u}(t,x,u^D) - \frac{\partial f}{\partial u}(t,x,u)\right)\cdot\nabla u^D.
\end{aligned}$$

Thus, for $(t,x)\in\Sigma_T$, one has (we recall that $u=u^D$):

$$\left(\frac{\partial f}{\partial u}(t,x,u^D) - \frac{\partial f}{\partial u}(t,x,u)\right) = 0,$$

and we obtain

$$\begin{aligned}
&\frac{\partial f}{\partial u}(t,x,u)\cdot\nabla v\\
&\qquad = \nabla\cdot\big(f(t,x,u)-f(t,x,u^D)\big) - \left\{(\nabla\cdot f)(t,x,u) - (\nabla\cdot f)(t,x,u^D)\right\}.
\end{aligned}$$

Since $\dfrac{\partial v}{\partial t} = 0$ on Σ_T, we have for $(t,x)\in\Sigma_T$

$$\begin{aligned}
&\frac{\partial f}{\partial u}(t,x,u)\cdot n\ z_j\ \frac{\partial \phi_\delta}{\partial \xi_j}(z) - \varepsilon\nabla z_i\cdot n\ \frac{\partial \phi_\delta}{\partial \xi_i}(z)\\
&= \left(\frac{\partial v}{\partial t} + \nabla\cdot\big(f(t,x,u)-f(t,x,u^D)\big) - \varepsilon\Delta v + \varepsilon\Delta s\nabla v\cdot n\right)\frac{\nabla v\cdot n}{(|\nabla v|^2+\delta^2)^{1/2}}\\
&\quad -\left\{\nabla\cdot f(t,x,u) - \nabla\cdot f(t,x,u^D)\right\}\frac{\nabla v\cdot n}{(|\nabla v|^2+\delta^2)^{1/2}}\\
&= (-h_1 + \varepsilon\Delta s\nabla v\cdot n)\,\frac{\nabla v\cdot n}{(|\nabla v|^2+\delta^2)^{1/2}}\\
&\quad -\left\{\nabla\cdot f(t,x,u) - \nabla\cdot f(t,x,u^D) + g(t,x,u) - g(t,x,u^D)\right\}\frac{\nabla v\cdot n}{(|\nabla v|^2+\delta^2)^{1/2}}\\
&\quad -\left\{g(t,x,u) - g(t,x,u^D)\right\}\frac{\nabla v\cdot n}{(|\nabla v|^2+\delta^2)^{1/2}}.
\end{aligned}$$

Since $(t,x) \in \Sigma_T$, we obtain $(u = u^D)$:

$$\frac{\partial f}{\partial u}(t,x,u)\cdot n\ z_j\ \frac{\partial \phi_\delta}{\partial \xi_j}(z) - \varepsilon \nabla z_i \cdot n\ \frac{\partial \phi_\delta}{\partial \xi_i}(z) = (-h_1 + \varepsilon \Delta s \nabla v \cdot n)\,\frac{\nabla v \cdot n}{(|\nabla v|^2 + \delta^2)^{1/2}}.$$

Putting this in the last inequality gives:

$$\int_\Omega \Big|z(t,\cdot)\Big| \quad \le \quad \int_\Omega \Big|z(0,\cdot)\Big| + \int_0^t \int_\Omega |h_2| + \int_0^t \int_{\partial\Omega} \Big\{ \Big|h_1\Big| + \varepsilon \mathcal{R} \Big|\nabla v \cdot n\Big| \Big\}$$

which, together with Inequality (3.33) implies

$$\int_\Omega \Big\{ \Big|\nabla v(t,\cdot)\Big| + \mathcal{R}\Big|v(t,\cdot)\Big| \Big\} \le \int_\Omega \Big\{ \Big|\nabla v(0,\cdot)\Big| + \mathcal{R}\Big|v(0,\cdot)\Big| \Big\} + \int_0^t \int_\Omega |h_2| + \int_0^t \int_{\partial\Omega} \Big|h_1\Big|$$

and, as a consequence,

$$\begin{aligned} \int_\Omega \Big\{ \Big|\nabla u(t,\cdot)\Big| + \mathcal{R}\Big|u(t,\cdot)\Big| \Big\} \quad \le \quad & \int_\Omega \Big\{ \Big|\nabla u^D(t,\cdot)\Big| + \mathcal{R}\Big|u^D(t,\cdot)\Big| \Big\} \\ & + \int_\Omega \Big\{ \Big|\nabla v(0,\cdot)\Big| + \mathcal{R}\Big|v(0,\cdot)\Big| \Big\} \\ & + \int_0^t \int_\Omega |h_2| + \int_0^t \int_{\partial\Omega} \Big|h_1\Big|. \end{aligned} \tag{3.34}$$

Let us analyse each term of Inequality (3.34).

► **(step $2^{(b)}$)** *Analysis of* $\int_\Omega \Big\{ \Big|\nabla u^D(t,\cdot)\Big| + \mathcal{R}\Big|u^D(t,\cdot)\Big| \Big\}$ - We easily state that

$$\int_\Omega \Big\{ \Big|\nabla u^D(t,\cdot)\Big| + \mathcal{R}\Big|u^D(t,\cdot)\Big| \Big\} \le c_6 = \left(\sup_{Q_T} \Big|\nabla u^D\Big| + \mathcal{R} \sup_{Q_T} \Big|u^D\Big| \right) |\Omega|\ T.$$

► **(step $2^{(b)}$)** *Analysis of* $\int_\Omega \Big\{ \Big|\nabla v(0,\cdot)\Big| + \mathcal{R}\Big|v(0,\cdot)\Big| \Big\}$ - Clearly, one has:

$$\begin{aligned} & \int_\Omega \Big\{ \Big|\nabla v(0,\cdot)\Big| + \mathcal{R}\Big|v(0,\cdot)\Big| \Big\} \\ & \qquad \le \int_\Omega \Big\{ \Big|\nabla u^0\Big| + \Big|\nabla u^D(0,\cdot)\Big| + \mathcal{R}\left(\Big|u^0\Big| + \Big|u^D(0,\cdot)\Big| \right) \Big| \Big\} \\ & \qquad \le \int_\Omega \left(\Big|\nabla u^0\Big| + \mathcal{R}\Big|u^0\Big| \right) + \left(\sup_{Q_T} \Big|\nabla u^D\Big| + \mathcal{R} \sup_{Q_T} \Big|u^D\Big| \right) = c_7. \end{aligned}$$

▶ **(step $2^{(b)}$)** *Analysis of* $\int_0^t \int_\Omega |h_1|$ - Recalling the expression of h_1,

$$\begin{aligned}\int_0^t \int_\Omega |h_1| &= \int_0^t \int_\Omega \Big| \frac{\partial u^D}{\partial t} + \nabla\cdot(f(t,x,u^D)) + g(t,x,u^D) - \varepsilon \Delta u^D \Big| \\ &\le \left(\sup_{Q_T} \Big| \frac{\partial u^D}{\partial t} + \Delta u^D \Big| + \sup_{Q_T\times[a,b]} |g| \right) \ |\Omega| \ T + \int_0^t \int_\Omega \nabla\cdot(f(t,x,u^D)).\end{aligned}$$

Since $\nabla\cdot(f(t,x,u^D)) = \nabla\cdot f(t,x,u^D) + \dfrac{\partial f}{\partial u}(t,x,u^D)\nabla u^D$, we have

$$\int_0^t \int_\Omega |h_1| \le c_8,$$

with $c_8 = \left(\sup_{Q_T} \left\{ \Big| \dfrac{\partial u^D}{\partial t} \Big| + \Big| \Delta u^D \Big| + \mathcal{L}_{[f]} \Big| \nabla u^D \Big| \right\} + \sup_{Q_T\times[a,b]} |g + \nabla\cdot f| \right) \ |\Omega| \ T.$

▶ **(step $2^{(b)}$)** *Analysis of* $\int_0^t \int_\Omega |h_2|$ - First, let us develop the expression of h_2:

$$\begin{array}{rlll} h_2^{(i)} = & \dfrac{\partial^2 u^D}{\partial x_i \partial t} & +\nabla\cdot \dfrac{\partial f}{\partial u}(t,x,u)\ \dfrac{\partial u^D}{\partial x_i} & +\dfrac{\partial^2 f}{\partial u^2}(t,x,u)\ \nabla u\ \dfrac{\partial u^D}{\partial x_i} \\ & +\dfrac{\partial f}{\partial u}(t,x,u)\ \nabla \dfrac{\partial u^D}{\partial x_i} & +\nabla\cdot \dfrac{\partial f}{\partial x_i}(t,x,u) & +\dfrac{\partial^2 f}{\partial x_i\, \partial u}(t,x,u)\ \nabla u \\ & +\dfrac{\partial g}{\partial x_i}(t,x,u) & +\dfrac{\partial g}{\partial u}(t,x,u)\ \dfrac{\partial u}{\partial x_i} & -\varepsilon \Delta \dfrac{\partial u^D}{\partial x_i}. \end{array}$$

From this, we get:

$$\int_0^t \int_\Omega |h_2| \le c_9^{(1)} + c_9^{(2)} \int_0^t \int_\Omega \Big| \nabla u \Big| \le c_9 \left(1 + \int_0^t \int_\Omega \Big| \nabla u \Big| \right)$$

with the following constants

$$\begin{aligned} c_9^{(1)} = & \left(\sup_{Q_T\times[a,b]} \Big| \nabla\cdot \frac{\partial f}{\partial u} \Big| \ \sup_{Q_T} \Big| \nabla u^D \Big| + \sup_{Q_T\times[a,b]} \Big| \nabla^2 f \Big| + \sup_{Q_T\times[a,b]} \Big| \nabla g \Big| \right) \ |\Omega| \ T \\ & + \mathcal{L}_{[f]} \int_{Q_T} \Big| \nabla^2 u^D \Big| + \int_{Q_T} \Big| \nabla^3 u^D \Big|, \end{aligned}$$

$$c_9^{(2)} = \sup_{Q_T\times[a,b]} \Big| \frac{\partial^2 f}{\partial u^2} \Big| \ \sup_{Q_T} \Big| \nabla u^D \Big| + \mathcal{L}_{[g]} + \sup_{Q_T\times[a,b]} \Big| \nabla\cdot \frac{\partial f}{\partial u} \Big|,$$

$$c_9 = \max\left(c_9^{(1)}, c_9^{(2)} \right).$$

To conclude **Step 2**, we gather Inequality (3.34) with all the previous bounds:

$$\int_\Omega \left\{ \left|\nabla u(t,\cdot)\right| + \mathcal{R}\left|u(t,\cdot)\right| \right\} \leq c_{10} \left(1 + \int_0^t \int_\Omega \left|\nabla u\right| \right)$$

where $c_{10} = c_6 + c_7 + c_8 + c_9$ does not depend on ε. Finally, since u is a function with values in $[a, b]$, from the previous inequality, we infer that there exists c_{11} which does not depend on ε such that

$$\int_\Omega \left|\nabla u(t,\cdot)\right| \leq c_{11} \left(1 + \int_0^t \int_\Omega \left|\nabla u\right| \right) \tag{3.35}$$

Now, we gather results obtained in **Step 1** and **Step 2**: using Inequalities (3.29) and (3.35) gives

$$\int_\Omega \left\{ \left|\frac{\partial u}{\partial t}(t,\cdot)\right| + \left|\nabla u(t,\cdot)\right| \right\} \leq c_{12} \left(1 + \int_0^t \int_\Omega \left\{ \left|\frac{\partial u}{\partial t}\right| + \left|\nabla u\right| \right\} \right) \tag{3.36}$$

where c_{12} ($= c_5 + c_{11}$, for instance) does not depend on ε. Applying Gronwall's lemma concludes the proof of Lemma 3.11. $\square$

Theorem 3.12 (Existence)**.** *Let us suppose that Assumption 13 holds. Let u_ε be the unique solution of Equations (3.18)–(3.20) corresponding to initial / boundary conditions $(u^0_\varepsilon, u^D_\varepsilon)$ satisfying Assumption 14 and let*

$$\begin{array}{rcll} \lim\limits_{\varepsilon\to 0} u^D_\varepsilon & = & u^D & \text{in } L^1(\Sigma_T), \\ \lim\limits_{\varepsilon\to 0} u^0_\varepsilon & = & u^0 & \text{in } L^1(\Omega), \end{array}$$

where $u^D \in L^\infty(\Sigma_T; [a, b])$ and $u^0 \in L^\infty(\Omega; [a, b])$. Then, the sequence $\{u_\varepsilon\}_\varepsilon$ converges to some function $u \in L^\infty(Q_T; [a, b])$ in $C^0([0, T], L^1(\Omega))$. Moreover u is a weak entroppy solution of problem (3.1)–(3.3).

Proof. Before entering into technical details, let us give the sketch of this proof. Our goal is to let ε tend to 0 in Equations (3.18)–(3.20). Nevertheless, we cannot apply estimates stated in Lemma 3.11 on u_ε because $u^D_\varepsilon, u^0_\varepsilon$ satisfy compatibility conditions but do not necessarily have an extension over $\overline{Q}_T$ with sufficient regularity. Thus, we introduce, by means of construction, $(u^D_{\varepsilon,h}, u^0_{\varepsilon,h})$ which both satisfy compatibility conditions and have an extension over $\overline{Q}_T$ with sufficient regularity. Moreover, $(u^D_{\varepsilon,h}, u^0_{\varepsilon,h})$ are uniformly "close" to $(u^D_\varepsilon, u^0_\varepsilon)$ (as $h \to 0$, uniformly with respect to ε), which implies that $u_{\varepsilon,h}$ is "close" to u_ε (in a sense which will be precised further). Then, we apply Arzelà-Ascoli theorem to the sequence $\{u_\varepsilon\}$ in order to prove that it is relatively compact in $C^0([0,T];L^1(\Omega))$. Of course, we have to verify that the sequence satisfies the hypotheses of the theorem (equicontinuity and pointwise relative compactness): for this, we use the properties of $u_{\varepsilon,h}$ and the fact that $u_{\varepsilon,h}$ is "close" to $u_{\varepsilon,h}$.

In order to use Lemma 3.11, we need some extension of u_ε^D and u_ε^0 to $\overline{Q}_T$, with sufficient regularity. Let us define the function $u_\varepsilon^{D,0}$ by

$$\begin{array}{rcll} u_\varepsilon^{D,0}(t,r+s\;n(r)) &=& u_\varepsilon^D(t,r), & t\in(0,T),\ r\in\partial\Omega,\ |s|\le\min(t,\delta) \\ u_\varepsilon^{D,0}(t,x) &=& u_\varepsilon^0(x), & -\delta<t<\min(\mathrm{dist}(x,\partial\Omega),\delta),\ x\in\Omega \\ u_\varepsilon^{D,0}(t,x) &=& 0, & \text{elsewhere.} \end{array}$$

Moreover, we mollify the earlier function (with a usual mollifier) which provides regularity on $\overline{Q}_T$:

$$u_{\varepsilon,h}^{D,0}(t,x)=\int_{\mathbb{R}^{d+1}} u_\varepsilon^{D,0}(t',x')\;\phi_h(t-t',x-x')\;dt'\;dx'.$$

Now we denote by $u_{\varepsilon,h}^D$ (resp. $u_{\varepsilon,h}^0$) the restriction of $u_{\varepsilon,h}^{D,0}$ to Σ_T (resp. $\{0\}\times\Omega$). Let $u_{\varepsilon,h}$ be the solution of Equations (3.18)–(3.20) corresponding to the boundary and initial conditions $u_{\varepsilon,h}^D$ and $u_{\varepsilon,h}^0$. On one hand, the uniform boundness of (u_ε^D, u_ε^0) implies the uniform boundness of ($u_{\varepsilon,h}^D$, $u_{\varepsilon,h}^0$) which provides (see Inequality (3.23) of Lemma 3.9) the uniform boundness of (u_ε, $u_{\varepsilon,h}$). Obviously, the following (strong) convergences hold:

$$\begin{array}{rcll} \lim\limits_{h\to0} u_{\varepsilon,h}^D &=& u_\varepsilon^D & \text{in } L^1(\Sigma_T) \\ \lim\limits_{h\to0} u_{\varepsilon,h}^0 &=& u_\varepsilon^0 & \text{in } L^1(\Omega) \end{array}$$

uniformly with respect to ε. This convergence result and Inequality (3.24) (see Lemma 3.10) imply

$$\lim_{h\to0} u_{\varepsilon,h}=u_\varepsilon \quad \text{in } C^0([0,T],L^1(\Omega)),$$

uniformly with respect to ε. On the other hand, it follows from the boundness of $u_\varepsilon^D\in L^1(\Sigma_T)$ and $u_\varepsilon^0\in L^1(\Omega)$ that

$$\left\|u_{\varepsilon,h}^D\right\|_{\Sigma_T}\le\frac{c}{h^3},\quad \left\|u_{\varepsilon,h}^0\right\|_{\Omega}\le\frac{c}{h^2}.$$

For fixed $h>0$ it follows from Inequality (3.25) that the sequences

$$\left\{\frac{\partial u_{\varepsilon,h}}{\partial t}\right\},\quad \{\nabla u_{\varepsilon,h}\}$$

are bounded in $C^0([0,T],L^1(\Omega))$. Now we propose to state that $\{u_\varepsilon\}_\varepsilon$ is precompact in

$C^0([0,T],L^1(\Omega))$ with the Arzelà-Ascoli theorem[4]:

(i) **Equicontinuity of $\{u_\varepsilon\}_\varepsilon$:** Let $\alpha>0$. Then there exists some $h>0$ such that

$$2\int_\Omega \Big|u_{\varepsilon,h}(t,\cdot)-u_\varepsilon(t,\cdot)\Big| < \alpha/2, \quad \forall t\in[0,T],\ \forall\varepsilon>0$$

and also, due to the uniform boundness with respect to ε of $\partial u_{\varepsilon,h}/\partial t$, there is some $\delta>0$ such that

$$\delta\int_\Omega \Big|\frac{\partial u_{\varepsilon,h}}{\partial t}(t,\cdot)\Big| < \alpha/2, \quad \forall t\in[0,T],\ \forall\varepsilon>0. \tag{3.37}$$

Thus, for all $\varepsilon>0$ and all t_1, $t_2\in[0,T]$ such that $|t_1-t_2|\le\delta$, we have

$$\begin{aligned}\int_\Omega \Big|u_\varepsilon(t_1,\cdot)-u_\varepsilon(t_2,\cdot)\Big| &\le \sum_{i=1}^{2}\int_\Omega \Big|u_{\varepsilon,h}(t_i,\cdot)-u_\varepsilon(t_i,\cdot)\Big| + \int_\Omega \Big|u_{\varepsilon,h}(t_1,\cdot)-u_{\varepsilon,h}(t_2,\cdot)\Big| \\ &\le \sum_{i=1}^{2}\int_\Omega \Big|u_{\varepsilon,h}(t_i,\cdot)-u_\varepsilon(t_i,\cdot)\Big| + \Big|t_1-t_2\Big|\sup_{t\in[t_1,t_2]}\int_\Omega \Big|\frac{\partial u_{\varepsilon,h}}{\partial t}(t,\cdot)\Big| \le \alpha. \end{aligned}\tag{3.38}$$

Thus, the sequence u_ε is equicontinuous in $C^0([0,T],L^1(\Omega))$.

(ii) **Pointwise relative compactness of $\{u_\varepsilon\}_\varepsilon$:** We use the Kolmogorov-Fréchet-Weil theorem[5]:

▷ Since $\{u_\varepsilon\}$ is uniformly bounded in $L^\infty(Q_T)$, $\{u_\varepsilon(t,\cdot)\}$ is also bounded in $L^1(\Omega)$

[4]**Arzelà-Ascoli theorem:** *Let (X,d_X) be a compact space, (Y,d_Y) a metric space. Then, a subset $\mathcal{H}$ of $C(X,Y)$ is relatively compact (for the the uniform convergence topology) if and only if $\mathcal{H}$ is equicontinuous and pointwise relatively compact.* We recall that

- $C(X,Y)$ denotes the set of all continuous functions from X to Y.
- The sequence $\{f_k\}$ is equicontinuous if for every $\alpha>0$ and every $x\in X$, there exists a $\delta>0$ such that for all k and all $x'\in X$ with $d_X(x,x')<\delta$ we have $d_Y(f_k(x),f_k(x'))<\alpha$.
- A subset $\mathcal{H}$ is pointwise relatively compact if and only if for all $x\in X$, the set $\{\hat h(x);\ \hat h\in\mathcal{H}\}$ is relatively compact in Y.

[5]**Kolmogorov-Fréchet-Weil theorem**: Let $1\le p<+\infty$ and $\Omega\subset\mathbb{R}^d$ (not necessarily bounded). A set $\mathcal{H}\subset L^p(\Omega)$ is relatively compact (for the strong topology) if and only if the following properties holds

▷ $\mathcal{H}$ is bounded

$$\sup_{f\in\mathcal{H}}\int_\Omega \Big|f\Big|^p < +\infty$$

▷ For all $\eta>0$, there exists $K_\eta\subset\Omega$ such that

$$\sup_{f\in\mathcal{H}}\int_{\Omega\setminus K_\eta}\Big|f\Big|^p < \eta$$

(uniformly with respect to $t \in [0,T]$ and ε).

▷ Let $\eta > 0$. Let us consider $K_\eta \subset \Omega$, defined by

$$K_\eta = \{x \in \Omega,\ \operatorname{dist}(x, \partial\Omega) \geq \eta\}.$$

Obviously, K_η is compact and

$$\sup_{u_\varepsilon(t,\cdot)} \int_{\Omega \setminus K_\eta} \left|u_\varepsilon(t,\cdot)\right| \leq \max(|a|,|b|) \operatorname{meas}(\Omega \setminus K_\eta) = C\eta,$$

where C only depends on $\partial\Omega$ and $\max(|a|,|b|)$.

▷ Recalling the existence of $\delta > 0$ such that Inequality (3.37) holds, we get uniformly in $t \in [0,T]$ and $\varepsilon > 0$,

$$\begin{aligned} \int_{\Omega^{\Delta_x}} & \left|u_\varepsilon(t, x+\Delta x) - u_\varepsilon(t,x)\right| \, dx \\ & \leq \ 2\int_\Omega \left|u_{\varepsilon,h}(t,\cdot) - u_\varepsilon(t,\cdot)\right| + |\Delta x| \int_\Omega \left|\nabla u_{\varepsilon,h}(t,\cdot)\right| \\ & < \ \alpha, \end{aligned}$$

for $|\Delta x| \leq \delta$ and $\Omega^{\Delta_x} = \{x \in \Omega,\ x + \Delta x \in \Omega\}$.

Thus, the sequence $\{u_\varepsilon(t,\cdot)\}_{t\in[0,T],\ \varepsilon>0}$ is relatively compact in $L^1(\Omega)$.

Thus, by the Arzelà-Ascoli theorem, $\{u_\varepsilon\}_\varepsilon$ is precompact in $C^0([0,T], L^1(\Omega))$, and since $C^0([0,T], L^1(\Omega))$ is complete, we infer[6] that, up to a subsequence,

$$\lim_{\varepsilon \to 0} u_\varepsilon = u \quad \text{in } C^0([0,T], L^1(\Omega)).$$

Finally, $u \in L^\infty(Q_T; [a,b])$ (see Inequality (3.23)). Now let us prove that u is a weak entropy solution of (3.1)–(3.3): recalling that the following properties hold:

$$\lim_{\varepsilon \to 0} \int_\Omega \left|1 - \xi_\varepsilon\right| = 0, \quad \lim_{\varepsilon \to 0} \varepsilon \int_\Omega \left|\nabla \xi_\varepsilon\right| = 0,$$

passing to the limit with respect to ε in Inequality (3.22) concludes the proof. □

▷ For all $\alpha > 0$, there exists $\delta > 0$ such that if $h < \delta$ then

$$\sup_{f \in \mathcal{H}} \int_{\Omega^h} \left|f(x+h) - f(x)\right|^p < \alpha$$

where Ω^h denotes the set $\{x \in \Omega; x + h \in \Omega\}$.

[6] A subspace of a complete metric space (Z, d_Z) is precompact if and only if every sequence admits a subsequence which converges in (Z, d_Z).

3.3 Uniqueness

Definition 8. *For any $k \in \mathbb{R}$, let us define the particular "boundary entropy-flux pairs"*[7]

$$\begin{array}{rccc} (\widetilde{H}^k, \widetilde{Q}^k_{[f]}) & : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & & (z,w) & \longrightarrow & \Big(\mathrm{dist}(z, \mathcal{I}(w,k)), \mathcal{F}_{[f]}(\cdot,\cdot,z,w,k)\Big) \end{array}$$

where $\mathcal{I}(w,k) = [\min(w,k), \max(w,k)]$ and $\mathcal{F}_{[f]} \in \mathcal{C}(\mathbb{R} \times \Omega \times \mathbb{R}^3)$ is defined as:

$$\mathcal{F}_{[f]}(\cdot,\cdot,z,w,k) = \begin{cases} f(\cdot,\cdot,w) - f(\cdot,\cdot,z), & \text{for} \quad z \le w \le k, \\ 0, & \text{for} \quad k \le z \le w, \\ f(\cdot,\cdot,z) - f(\cdot,\cdot,k), & \text{for} \quad w \le k \le z, \\ f(\cdot,\cdot,k) - f(\cdot,\cdot,z), & \text{for} \quad z \le k \le w, \\ 0, & \text{for} \quad w \le z \le k, \\ f(\cdot,\cdot,z) - f(\cdot,\cdot,w), & \text{for} \quad k \le w \le z. \end{cases}$$

Lemma 3.13. *Let $u \in L^\infty(Q_T)$ satisfy $(\mathcal{P}_{SK})$; then one has:*

- *for all $\varphi \in \mathcal{D}((0,T) \times \mathbb{R}^d)$, for all $k \in \mathbb{R}$,*

$$\begin{aligned} \int_{Q_T} \Bigg\{ & |u-k| \frac{\partial \varphi}{\partial t} + \mathrm{sgn}(u-k)\Big(f(t,x,u) - f(t,x,k)\Big) \nabla\varphi \\ & -\mathrm{sgn}(u-k)\left(\nabla \cdot f(t,x,k) + g(t,x,u)\right) \varphi \Bigg\} \, dx \, dt \\ & \ge \operatorname*{ess\,lim}_{\varrho \to 0^+} \int_{\Sigma_T} \Bigg\{ \mathrm{sgn}(u(t, r - \varrho\, n(r)) - k)\Big(f(t, r-\varrho\, n(r), u(t, r - \varrho\, n(r))) \\ & \qquad - f(t, r - \varrho\, n(r), k)\Big) \Bigg\} \cdot n(r)\, \varphi(t,r)\, d\gamma(r)\, dt, \end{aligned}$$

- *for all $\beta \in L^1(\Sigma_T)$, $\beta \ge 0$ a.e., and for all $k \in \mathbb{R}$,*

$$\operatorname*{ess\,lim}_{\varrho \to 0^+} \int_{\Sigma_T} \mathcal{F}_{[f]}(t, r, u(t, r - \varrho\, n(r)), u^D(t,r), k) \cdot n(r)\, \beta(t,r)\, d\gamma(r)\, dt \ge 0,$$

- *for all $\varphi \in \mathcal{D}((0,T) \times \mathbb{R}^d)$, for all $k \in \mathbb{R}$,*

[7] Although $(\widetilde{H}^k, \widetilde{Q}^k_{[f]})$ has less regularity than required in Definition 6, it is obtained (see Remark 7.8 in [MNRR96]) by uniform convergence, as $\delta \to 0$, of the sequence of "boundary entropy-flux pairs"

$$\begin{aligned} \widetilde{H}^{k,\delta}(z,w) &= \Big(\mathrm{dist}(z, \mathcal{I}(w,k))^2 - \delta^2\Big)^{1/2} - \delta, \\ \widetilde{Q}^{k,\delta}_{[f]}(\cdot,\cdot,z,w) &= \int_w^z \partial_1 \widetilde{H}^\delta_k(\lambda, w) \frac{\partial f}{\partial u}(\cdot,\cdot,\lambda)\, d\lambda \end{aligned}$$

$$\begin{aligned}\int_{Q_T} \Big\{ & |u-k| \frac{\partial \varphi}{\partial t} + \operatorname{sgn}(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\ \nabla\varphi \\ & -\operatorname{sgn}(u-k)\left(\nabla\cdot f(t,x,k)+g(t,x,u)\right)\ \varphi\Big\}\ dx\ dt \\ & \geq \int_{\Sigma_T} \operatorname{sgn}(k-u^D)\Big(f(t,r,k)-f(t,r,u^D)\Big)\cdot n(r)\ \varphi(t,r)\ d\gamma(r)\ dt \\ & -\operatorname{ess}\lim_{\varrho\to 0^+}\int_{\Sigma_T}\Big\{\operatorname{sgn}(u(t,r-\varrho\ n(r))-k)\Big(f(t,r,u(t,r-\varrho\ n(r))) \\ & \qquad -f(t,r,k)\Big)\Big\}\cdot n(r)\ \varphi(t,r)\ d\gamma(r)\ dt.\end{aligned}$$

Proof.

▷ *1st inequality* - Adding the two inequalities defined by $(\mathcal{P}_{SK})$ with each "semi Kružkov entropy-flux pair" gives the following inequality

$$\begin{aligned}\int_{Q_T} \Big\{ & |u-k| \frac{\partial \varphi}{\partial t} + \operatorname{sgn}(u-k)\Big(f(t,x,u)-f(t,x,k)\Big)\ \nabla\varphi \\ & -\operatorname{sgn}(u-k)\left(\nabla\cdot f(t,x,k)+g(t,x,u)\right)\ \varphi\Big\}\ dx\ dt \geq 0,\end{aligned} \tag{3.39}$$

for any $\varphi \in \mathcal{D}(Q_T)$. Thus, since u satisfies Inequality (3.39) along with the initial condition (3.9) (see Lemma 3.4), the result is obtained by following the same lines of the proof of Lemma 7.12. in [MNRR96].

▷ *2nd inequality* - The result is easily obtained by Lemma 3.3 applied to the particular "boundary fluxes" $\mathcal{F}_{[f]}$ (see Definition 8).

▷ *3rd inequality* - On the one hand, the function $2\mathcal{F}_{[f]}(\cdot,\cdot,z,w,k)$ is equal to

$$\begin{aligned}\operatorname{sgn}(z-w)\ \ \Big(f(\cdot,\cdot,z)-f(\cdot,\cdot,w)\Big)\ \ -\ \ & \operatorname{sgn}(k-w)\ \ \Big(f(\cdot,\cdot,k)-f(\cdot,\cdot,w)\Big) \\ +\ \ & \operatorname{sgn}(z-k)\ \ \Big(f(\cdot,\cdot,z)-f(\cdot,\cdot,k)\Big).\end{aligned}$$

On the second hand, terms of the form

$$\begin{aligned}\operatorname{ess}\lim_{\varrho\to 0^-}\int_{\Sigma_T} & \operatorname{sgn}(u(t,r-\varrho\ n)-v^D(t,r))\Big\{f(t,r,u(t,r-\varrho\ n)) \\ & -f(t,r,v^D(t,r))\Big\}\cdot n\ \beta(t,r)\ d\gamma(r)\ dt\end{aligned}$$

exist for all $\beta \in L^1(\Sigma_T)$, all $v^D \in L^\infty(\Sigma_T)$. Indeed, this term is obtained by using the proof of Lemma 3.3: it is sufficient to add the terms of (3.8) corresponding to each "semi

Kružkov entropy-flux pair". Therefore, each term of the following inequality exists. Thus, from the 2nd inequality,

$$\begin{aligned}
&-\operatorname{ess}\lim_{\varrho\to 0^+}\int_{\Sigma_T} F(t,r,u(t,r-\varrho\, n(r)),k)\cdot n(r)\ \beta(t,r)\ d\gamma(r)\ dt\\
&\leq \operatorname{ess}\lim_{\varrho\to 0^+}\int_{\Sigma_T} F(t,r,u(t,r-\varrho\, n(r)),u^D(t,r))\cdot n(r)\ \beta(t,r)\ d\gamma(r)\ dt\\
&\qquad\qquad -\int_{\Sigma_T} F(t,r,k,u^D(t,r))\cdot n(r)\ \beta(t,r)\ d\gamma(r)\ dt
\end{aligned}$$

with the notation $F(t,r,u,k)=\operatorname{sgn}(u-k)\Big(f(t,r,u)-f(t,r,k)\Big)$, and the result is straightforward. □

Lemma 3.14. *Let $u\in L^\infty(Q_T)$ (resp. $v\in L^\infty(Q_T)$) be a solution of $(\mathcal{P}_{SK})$ with initial and boundary conditions $(u^0,u^D)\in L^\infty(\Omega)\times L^\infty(\Sigma_T)$ (resp. $(v^0,v^D)\in L^\infty(\Omega)\times L^\infty(\Sigma_T)$); then*

$$\begin{aligned}
&-\int_{Q_T}\Bigg\{\Big|u-v\Big|\ \frac{\partial\beta}{\partial t}\ +\ \operatorname{sgn}(u-v)\Big(f(t,x,u)-f(t,x,v)\Big)\nabla\beta\\
&\qquad\qquad -\ \operatorname{sgn}(u-v)\left(g(t,x,u)-g(t,x,v)\right)\beta\Bigg\}dx\ dt\\
&\leq \int_\Omega \Big|u^0(x)-v^0(x)\Big|\ \beta(0,x)\ dx+\mathcal{L}_{[f]}\int_{\Sigma_T}\Big|u^D(t,r)-v^D(t,r)\Big|\ \beta(t,r)\ d\gamma(r)\ dt
\end{aligned}$$

for all $\beta\in\mathcal{D}((-\infty,T)\times\mathbb{R}^d)$.

Proof. As it was already pointed out, each term that can be written under the form

$$\begin{aligned}
\operatorname{ess}\lim_{\varrho\to 0^+}\int_{\Sigma_T}\Bigg\{&\operatorname{sgn}(u(t,r-\varrho\, n(r))-v^D(t,r))\Big(f(t,r,u(t,r-\varrho\, n(r)))\\
&-f(t,r,v^D(t,r))\Big)\Bigg\}\cdot n(r)\ \beta(t,r)\ d\gamma(r)\ dt
\end{aligned}$$

exists for all $\beta\in L^1(\Sigma_T)$, all $v^D\in L^\infty(\Sigma_T)$. Thus, we infer that there exists $\theta_{i,j}\in L^\infty(\Sigma_T)$ such that:

$$\int_{\Sigma_T} \theta_{1,1}(t,r)\ \beta(t,r)\ d\gamma(r)\ dt =$$
$$\operatorname{ess}\lim_{\varrho\to 0^+} \int_{\Sigma_T} \operatorname{sgn}(u(t,r-\varrho\ n)-u^D)\Big(f(t,r,u(t,r-\varrho\ n))-f(t,r,u^D)\Big)\cdot n\ \beta\ d\gamma(r)\ dt,$$

$$\int_{\Sigma_T} \theta_{1,2}(t,r)\ \beta(t,r)\ d\gamma(r)\ dt =$$
$$\operatorname{ess}\lim_{\varrho\to 0^+} \int_{\Sigma_T} \operatorname{sgn}(u(t,r-\varrho\ n)-v^D)\Big(f(t,r,u(t,r-\varrho\ n))-f(t,r,v^D)\Big)\cdot n\ \beta\ d\gamma(r)\ dt,$$

$$\int_{\Sigma_T} \theta_{2,2}(t,r)\ \beta(t,r)\ d\gamma(r)\ dt =$$
$$\operatorname{ess}\lim_{\varrho\to 0^+} \int_{\Sigma_T} \operatorname{sgn}(v(t,r-\varrho\ n)-v^D)\Big(f(t,r,v(t,r-\varrho\ n))-f(t,r,v^D)\Big)\cdot n\ \beta\ d\gamma(r)\ dt,$$

$$\int_{\Sigma_T} \theta_{2,1}(t,r)\ \beta(t,r)\ d\gamma(r)\ dt =$$
$$\operatorname{ess}\lim_{\varrho\to 0^+} \int_{\Sigma_T} \operatorname{sgn}(v(t,r-\varrho\ n)-u^D)\Big(f(t,r,v(t,r-\varrho\ n))-f(t,r,u^D)\Big)\cdot n\ \beta\ d\gamma(r)\ dt.$$

After this introduction of notations, we now apply the double variable method, initiated by Kružkov [Kru70], to the 3rd inequality stated in Lemma 3.13. Let $\rho_\varepsilon \in \mathcal{D}(\mathbb{R}^{d+1})$ be a symmetric regularizing sequence. For the sake of simplicity, we denote

$$\begin{array}{rclrcl} p &=& (t,x)\in Q_T, & p' &=& (t',x')\in Q_T,\\ \gamma(p) &=& (t,r)\in\Sigma_T, & \gamma(p') &=& (t',r')\in\Sigma_T, \end{array}$$

and let

$$\beta_\varepsilon(p,p') = \beta\left(\frac{p+p'}{2}\right)\rho_\varepsilon\left(p-p'\right),$$

for all $p,p'\in(Q_T)^2$, for a given $\beta\in\mathcal{D}((0,T)\times\mathbb{R}^d)$, $\beta\geq 0$. Hold $p'\in Q_T$ fixed and replace, in the 3rd inequality of Lemma 3.13, k by $v(p')$ and $\beta(p)$ by $\beta_\varepsilon(p,p')$. After integration over Q_T (with respect to the variable p'), and using the notation

$$F(p,u(p),v(p')) = \operatorname{sgn}(u(p)-v(p'))\left(f(p,u(p)-f(p,v(p')))\right),$$

we easily get $I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon + I_4^\varepsilon + I_5^\varepsilon \le I_6^\varepsilon + I_7^\varepsilon$, with

$$\begin{aligned}
I_1^\varepsilon &= -\frac{1}{2}\int_{Q_T}\int_{Q_T} \left|u(p) - v(p')\right| \frac{\partial \beta}{\partial t}\left(\frac{p+p'}{2}\right)\ \rho_\varepsilon\left(p-p'\right) dp\ dp' \\
I_2^\varepsilon &= -\frac{1}{2}\int_{Q_T}\int_{Q_T} F(p,u(p),v(p'))\ \nabla\beta\left(\frac{p+p'}{2}\right)\ \rho_\varepsilon\left(p-p'\right) dp\ dp' \\
I_3^\varepsilon &= -\int_{Q_T}\int_{Q_T} \left|u(p) - v(p')\right| \frac{\partial \rho_\varepsilon}{\partial t}\left(p-p'\right)\ \beta\left(\frac{p+p'}{2}\right) dp\ dp' \\
I_4^\varepsilon &= -\int_{Q_T}\int_{Q_T} F(p,u(p),v(p'))\ \nabla\rho_\varepsilon\left(p-p'\right)\ \beta\left(\frac{p+p'}{2}\right) dp\ dp' \\
I_5^\varepsilon &= \int_{Q_T}\int_{Q_T} \operatorname{sgn}(u(p)-v(p'))\left\{\nabla\cdot f\left(p,v(p')\right) + g(p,u(p))\right\}\ \beta\left(\frac{p+p'}{2}\right)\ \rho_\varepsilon\left(p-p'\right)\ dp\ dp' \\
I_6^\varepsilon &= \int_{Q_T}\int_{\Sigma_T} \theta_{1,1}(\gamma(p))\ \beta\left(\frac{\gamma(p)+p'}{2}\right)\ \rho_\varepsilon\left(\gamma(p)-p'\right)\ d\gamma(p)\ dp' \\
I_7^\varepsilon &= -\int_{Q_T}\int_{\Sigma_T} F(p,v(p'),u^D(\gamma(p)))\cdot n\ \beta\left(\frac{\gamma(p)+p'}{2}\right)\ \rho_\varepsilon\left(\gamma(p)-p'\right)\ d\gamma(p)\ dp'.
\end{aligned}$$

Now changing the role of $(u(p),p)$ and $(v(p'),p')$, we get similarly

$$J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon + J_4^\varepsilon + J_5^\varepsilon \le J_6^\varepsilon + J_7^\varepsilon,$$

with

$$\begin{aligned}
J_1^\varepsilon &= -\frac{1}{2}\int_{Q_T}\int_{Q_T} \left|v(p') - u(p)\right| \frac{\partial \beta}{\partial t}\left(\frac{p+p'}{2}\right)\ \rho_\varepsilon\left(p-p'\right) dp\ dp' \\
J_2^\varepsilon &= -\frac{1}{2}\int_{Q_T}\int_{Q_T} F(p',v(p'),u(p))\ \nabla\beta\left(\frac{p+p'}{2}\right)\ \rho_\varepsilon\left(p-p'\right) dp\ dp' \\
J_3^\varepsilon &= \int_{Q_T}\int_{Q_T} \left|v(p') - u(p)\right| \frac{\partial \rho_\varepsilon}{\partial t}\left(p-p'\right)\ \beta\left(\frac{p+p'}{2}\right) dp\ dp' \\
J_4^\varepsilon &= \int_{Q_T}\int_{Q_T} F(p',v(p'),u(p))\ \nabla\rho_\varepsilon\left(p-p'\right)\ \beta\left(\frac{p+p'}{2}\right) dp\ dp' \\
J_5^\varepsilon &= \int_{Q_T}\int_{Q_T} \operatorname{sgn}(v(p')-u(p))\left\{\nabla\cdot f\left(p',u(p)\right) + g(p',v(p'))\right\}\ \beta\left(\frac{p+p'}{2}\right)\ \rho_\varepsilon\left(p-p'\right)\ dp\ dp' \\
J_6^\varepsilon &= \int_{Q_T}\int_{\Sigma_T} \theta_{2,2}(\gamma(p'))\ \beta\left(\frac{p+\gamma(p')}{2}\right)\ \rho_\varepsilon\left(p-\gamma(p')\right)\ d\gamma(p')\ dp \\
J_7^\varepsilon &= -\int_{Q_T}\int_{\Sigma_T} F(p',u(p),v^D(\gamma(p')))\cdot n\ \beta\left(\frac{p+\gamma(p')}{2}\right)\ \rho_\varepsilon\left(p-\gamma(p')\right)\ d\gamma(p')\ dp.
\end{aligned}$$

Adding the two inequalities, let us remark that

$$\begin{aligned}
I_1^\varepsilon &= J_1^\varepsilon, \\
I_3^\varepsilon &= -J_3^\varepsilon,
\end{aligned}$$

so that we have

$$2I_1^\varepsilon + (I_2^\varepsilon + J_2^\varepsilon) + (I_4^\varepsilon + J_4^\varepsilon) + (I_5^\varepsilon + J_5^\varepsilon) \le (I_6^\varepsilon + J_6^\varepsilon) + (I_7^\varepsilon + J_7^\varepsilon).$$

We are now ready to pass to the limit on ε: for convenience, proofs are omitted. Let us mention that this method has been widely used in the works related to hyperbolic problems [Kru70, BLRN79, Ott96, MNRR96] but also parabolic problems [Car99] or elliptic-hyperbolic problems (in free boundary problems applied to lubrication theory, [AC94, AO03, Váz94] and also Chapter 1 of this thesis). Thus,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} I_1^\varepsilon = -\frac{1}{2}\int_{Q_T} |u-v|\ \frac{\partial \beta}{\partial t},\\
&\lim_{\varepsilon\to 0} I_2^\varepsilon = \lim_{\varepsilon\to 0} J_2^\varepsilon = -\frac{1}{2}\int_{Q_T} \operatorname{sgn}(u-v)\ \ (f(t,x,u)-f(t,x,v)))\ \ \nabla\beta,\\
&\lim_{\varepsilon\to 0}(I_4^\varepsilon + J_4^\varepsilon + I_5^\varepsilon + J_5^\varepsilon) = \int_{Q_T} \operatorname{sgn}(u-v)\,(g(t,x,u)-g(t,x,v))\,\beta,\\
&\lim_{\varepsilon\to 0} I_6^\varepsilon = \frac{1}{2}\int_{\Sigma_T} \theta_{1,1}\ \beta, \quad \lim_{\varepsilon\to 0} J_6^\varepsilon = \frac{1}{2}\int_{\Sigma_T} \theta_{2,2}\ \beta,\\
&\lim_{\varepsilon\to 0} I_7^\varepsilon = -\frac{1}{2}\int_{\Sigma_T} \theta_{1,2}\ \beta, \quad \lim_{\varepsilon\to 0} J_7^\varepsilon = -\frac{1}{2}\int_{\Sigma_T} \theta_{2,1}\ \beta.
\end{aligned}$$

Finally we obtain:

$$\begin{aligned}
-\int_{Q_T}\Bigg\{\Big|u-v\Big|\ \frac{\partial\beta}{\partial t} \quad &+ \quad \operatorname{sgn}(u-v)\Big(f(t,x,u)-f(t,x,v)\Big)\nabla\beta\\
&- \quad \operatorname{sgn}(u-v)\,(g(t,x,u)-g(t,x,v))\,\beta\Bigg\}\\
\le \frac{1}{2}\int_{\Sigma_T} (-\theta_{1,1}+\theta_{2,1}-\theta_{2,2}+\theta_{1,2})\,\beta&,
\end{aligned}$$

for all $\beta \in \mathcal{D}((0,T)\times\mathbb{R}^d)$. As in [MNRR96], let us introduce the following definition:

$$\forall (t,r)\in\Sigma_T,\ \operatorname{diam}\Big(f(t,r,\cdot)\cdot n, \mathcal{I}(a,b)\Big) = \sup_{z_1,z_2\in\mathcal{I}(a,b)} \Big(\Big|f(t,r,z_1)\cdot n - f(t,r,z_2)\cdot n\Big|\Big)$$

Then, if one discusses the cases, one sees that for all $z_1, z_2, w_1, w_2 \in \mathbb{R}$, the inequalities

$$\Big|\sum_{i,j=1}^{2}(-1)^{i+j}\operatorname{sgn}(z_i-w_j)\Big(f(t,r,z_i)-f(t,r,w_j)\Big)\cdot n\Big| \le 2\ \operatorname{diam}(f(t,r,\cdot)\cdot n, \mathcal{I}(w_1,w_2))$$

hold and using the property

$$\operatorname{diam}(f(t,r,\cdot)\cdot n, \mathcal{I}(u^D(t,r),v^D(t,r))) \le \mathcal{L}_{[f]}\Big|u^D(t,r)-v^D(t,r)\Big|, \quad \forall (t,r)\in\Sigma_T,$$

one easily concludes that

$$\frac{1}{2}\Big|\int_{\Sigma_T} (-\theta_{1,1}+\theta_{2,1}-\theta_{2,2}+\theta_{1,2})\ \beta\Big| \leq \mathcal{L}_{[f]}\int_{\Sigma_T}\Big|u^D - v^D\Big|\ \beta.$$

The initial term is obtained by slightly modifying the proof, with test functions in the appropriate space, namely $\mathcal{D}((-\infty,T)\times\mathbb{R}^d)$. □

Theorem 3.15 (Uniqueness)**.** *Under Assumption 13, problem* $(\mathcal{P}_{SK})$ *admits a unique weak entropy solution.*

Proof. Considering the integral inequality of Lemma 3.14 with $v^D = u^D$ and $v^0 = u^0$ and a test function which only depends on time t, we get:

$$\int_{Q_T}\left\{\Big|u-v\Big|\ \alpha'(t) - \mathrm{sgn}(u-v)\Big(g(t,x,u)-g(t,x,v)\Big)\ \alpha(t)\right\}dx\,dt \geq 0, \tag{3.40}$$

for all $\alpha \in \mathcal{D}(-\infty,T)$. Then, for an interval $[t_0,t_1]\ \subset]0,T[$, we can use in Inequality (3.40) the characteristic function of $[t_0,t_1]$, properly mollified, and pass to the limit on the mollifier parameter:

$$\begin{aligned}\int_\Omega \Big|u(t_1,\cdot)-v(t_1,\cdot)\Big| \ &\leq\ \int_\Omega\Big|u(t_0,\cdot)-v(t_0,\cdot)\Big| \\ &+\int_{t_0}^{t_1}\int_\Omega \mathrm{sgn}(u-v)\,(g(t,x,v)-g(t,x,u))\ dx\ dt.\end{aligned}$$

Now, since we have, for all $(t,x)\in(0,T)\times\Omega$:

$$\mathrm{sgn}(u-v)\,(g(t,x,v)-g(t,x,u)) \leq \mathcal{L}_{[g]}\Big|u-v\Big|,$$

where $\mathcal{L}_{[g]}$ is the Lipschitz constant of continuity with respect to u of g, we obtain:

$$\int_\Omega\Big|u(t_1,\cdot)-v(t_1,\cdot)\Big| \leq \int_\Omega\Big|u(t_0,\cdot)-v(t_0,\cdot)\Big| + \mathcal{L}_{[g+\nabla\cdot f]}\int_{t_0}^{t_1}\Big|u(t,\cdot)-v(t,\cdot)\Big|_{L^1(\Omega)}dt.$$

From Gronwall's lemma, we conclude that:

$$\Big|u(t_1,\cdot)-v(t_1,\cdot)\Big|_{L^1(\Omega)} \leq \Big|u(t_0,\cdot)-v(t_0,\cdot)\Big|_{L^1(\Omega)} e^{\mathcal{L}_{[g]}(t_1-t_0)}.$$

As t_0 tends to 0, and using the fact that $v^0 = u^0$ along with the initial condition (3.9), the uniqueness is straightforward. □

Remark 3.16. *Let us conclude with the results obtained in the case of the physical problems that we have introduced.*

- ***Lubrication theory -*** *The generalized lubrication problem (Equation (3.4) with its corresponding initial and boundary conditions s^0, s^D) admits a unique weak entropy solution $s \in L^\infty(Q_T)$ in the following sense:*

$$\int_0^L \int_0^T \Big\{ h(x)\ (s-k)^\pm \frac{\partial \varphi}{\partial t}$$
$$+\mathrm{sgn}_\pm(s-k)\ \Big(Q_{in}(t)\ (f_1(s)-f_1(k)) + v_0 h(x)\ (f_2(s)-f_2(k)) \Big) \frac{\partial \varphi}{\partial x}$$
$$-\mathrm{sgn}_\pm(s-k)\ v_0\ h'(x)\ f_2(k)\ \varphi \Big\}\ dx\ dt + \int_0^L (s^0-k)^\pm \varphi(0,x)\ dx$$
$$+\mathcal{L}_{[f]} \int_0^T (s^D(t,0)-k)^\pm \varphi(t,0)\ dt + \mathcal{L}_{[f]} \int_0^T (s^D(t,1)-k)^\pm \varphi(t,1)\ dt \geq 0$$

$\forall \phi \in \mathcal{D}((-\infty,T)\times\mathbb{R})$, $\phi \geq 0$, $\forall k \in \mathbb{R}$.

Here, $\mathcal{L}_{[f]} = \max(Q_{in})\mathcal{L}_{[f_1]} + v_0\ \max(h)\ \dfrac{\max(Q_{in})}{\min(Q_{in})}\ \mathcal{L}_{[f_2]}$. *Moreover, s is a function with values in the set* $[0,1]$.

- ***Environmental sciences -*** *The transport problem (Equation (3.6) with the corresponding homogeneous initial and boundary conditions $c^0 \equiv 0$, $c^D \equiv 0$) admits a unique weak entropy solution $c \in L^\infty(Q_T)$ in the following sense:*

$$\int_{Q_T} \Big\{ (c-k)^\pm \frac{\partial \varphi}{\partial t} \quad + \quad (c-k)^\pm\ U(t,x)\ \nabla\varphi$$
$$- \quad \mathrm{sgn}_\pm(c-k)\ \frac{1}{h}(ic - J_b)\ \varphi \Big\}\ dx\ dt$$
$$+ \quad \int_\Omega (c^0-k)^\pm \varphi(0,x)\ dx$$
$$+ \quad \mathcal{L}_{[f]} \int_{\Sigma_T} (c^D-k)^\pm \varphi(t,r)\ d\gamma(r)\ dt \geq 0$$

$\forall \phi \in \mathcal{D}((-\infty,T)\times\mathbb{R}^2)$, $\phi \geq 0$, $\forall k \in \mathbb{R}$.

with, for instance, $\mathcal{L}_{[f]} = \|U\|_{L^\infty}$. *The analysis also provides a critical (worst) value for the concentration, which satisfies* $0 \leq c \leq J_b/i$.

Bibliography

[AC94] S. J. Alvarez and J. Carrillo. A free boundary problem in theory of lubrication. *Comm. Partial Differential Equations*, 19(11-12):1743–1761, 1994.

[AO03] S. J. Alvarez and R. Oujja. On the uniqueness of the solution of an evolution free boundary problem in theory of lubrication. *Nonlinear Anal.*, 54(5):845–872, 2003.

[AS79] K. Aziz and A. Settari. *Petroleum reservoir simulation.* Applied Science Publishers, London, 1979.

[BKT05] R. Bürger, K. H. Karlsen, and J. D. Towers. A model of continuous sedimentation of flocculated suspensions in clarifier-thickener units. *SIAM J. Appl. Math.*, 65(3):882–940, 2005.

[BLRN79] C. Bardos, A. Y. Le Roux, and J.-C. Nédélec. First order quasilinear equations with boundary conditions. *Comm. Partial Differential Equations*, 4(9):1017–1034, 1979.

[BMV05] G. Bayada, S. Martin, and C. Vázquez. About a generalized Buckley-Leverett equation and lubrication multifluid flow. *Preprint Institut Camille Jordan CNRS-UMR 5208 (submitted for publication)*, 2005.

[Car99] J. Carrillo. Entropy solutions for nonlinear degenerate problems. *Arch. Ration. Mech. Anal.*, 147(4):269–361, 1999.

[CJ86] G. Chavent and J. Jaffre. *Mathematical Methods and Finite Elements for Reservoir Simulation*, volume 17 of *Studies in Mathematics and its Applications*. North-Holland, Amsterdam, 1986.

[DL88] F. Dubois and P. LeFloch. Boundary conditions for nonlinear hyperbolic systems of conservation laws. *J. Differential Equations*, 71(1):93–122, 1988.

[GMT96] G. Gagneux and M. Madaune-Tort. *Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière,*, volume 22 of *Mathématiques & Applications [Mathematics & Applications]*. Springer-Verlag, Berlin, 1996.

[Kaa99] E. F. Kaasschieter. Solving the Buckley-Leverett equation with gravity in a heterogeneous porous medium. *Comput. Geosci.*, 3(1):23–48, 1999.

[Kru70] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[MNRR96] J. Málek, J. Nečas, M. Rokyta, and M. Ružička. *Weak and measure-valued solutions to evolutionary PDEs*, volume 13 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1996.

[Ole59] O. A. Oleĭnik. Uniqueness and stability of the generalized solution of the Cauchy problem for a quasi-linear equation. *Uspehi Mat. Nauk*, 14(2 (86)):165–170, 1959.

[Ott96] F. Otto. Initial-boundary value problem for a scalar conservation law. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(8):729–734, 1996.

[Pao03] L. Paoli. Asymptotic behavior of a two fluid flow in a thin domain: from Stokes equations to Buckley-Leverett equation and Reynolds law. *Asymptot. Anal.*, 34(2):93–120, 2003.

[Sak64] S. Saks. *Theory of the integral.* Second revised edition. English translation by L. C. Young. With two additional notes by Stefan Banach. Dover Publications Inc., New York, 1964.

[Ser96] D. Serre. *Systèmes de lois de conservation. II.* Fondations. [Foundations]. Diderot Editeur, Paris, 1996. Structures géométriques, oscillation et problèmes mixtes. [Geometric structures, oscillation and mixed problems].

[Váz94] C. Vázquez. Existence and uniqueness of solution for a lubrication problem with cavitation in a journal bearing with axial supply. *Adv. Math. Sci. Appl.*, 4(2):313–331, 1994.

[Vol67] A. I. Volpert. Spaces BV and quasilinear equations. *Mat. Sb. (N.S.)*, 73 (115):255–302, 1967.

[Vov02] J. Vovelle. Convergence of finite volume monotone schemes for scalar conservation laws on bounded domains. *Numer. Math.*, 90(3):563–596, 2002.

[WPW+01] M.T. Walter, J.-Y. Parlange, M.F. Walter, X. Xin, and C.A. Scott. Modeling pollutant release from a surface source during rainfall-runoff. *J. Env. Qual.*, 30(1):151–159, 2001.

4

About a generalized Buckley-Leverett equation and lubrication multifluid flow

Article soumis pour publication.

Abstract *In this chapter, we analyse the asymptotic system corresponding to a thin film flow with two different fluids, from theoretical and numerical point of view. We also compare this model to the Elrod-Adams one, which is the reference model in tribology, as cavitation phenomena occur. In this way, the Elrod-Adams model may be justified by the bifluid approach in which a liquid/gas mixture is considered.*

4.0 Introduction

The asymptotic behaviour of a single incompressible flow between two close surfaces in relative motion is described by the well-known Reynolds equation

$$\nabla \cdot \Big(\frac{h^3}{6\mu_l}\nabla p\Big) = v_0 \frac{\partial}{\partial x_1}\Big(h\Big),$$

in which h is the small gap between the two surfaces, v_0 the relative velocity of the surfaces, μ_l the fluid viscosity and p the fluid pressure. This equation can be rigorously deduced from (Navier) Stokes system by means of an asymptotic analysis [BC86]. However, in some applications, the lubricant cannot be considered as a single fluid and a multifluid approach has to be introduced. For example, this happens when one of the surfaces has to be particularly protected from contact from the other one and it is covered by a specific fluid; this can be also modelled by the existence of a surface layer with a viscosity which is different from the one of the bulk fluid. Another phenomenon which falls into the scope of the multifluid approach is linked to cavitation, which introduces the existence of air bubbles inside the bulk fluid.

Nevertheless, most of the multifluid problems in the lubrication area assume that the boundary between the two immiscible fluids is known [BCG01, SJPT91, Tic96]. This assumption allows to obtain a slightly modified Reynolds equation. However, the real problem is a free boundary one and the position of the interface is defined by an additional unknown function. Using a three dimensional multifluid approach introduced by Nouri, Poupaud and Demay [NPD97], a limit system has been derived by Bayada, Sabil and Paoli [Sab00, Pao03] with an asymptotic approach. This system describes the behaviour of the pressure and the relative saturation of the two fluids contained in a thin domain. However, this derivation is based upon an assumption on the shape of the interface. Thus, a mathematical study of the obtained asymptotic system has to be made in order to ensure its well-posedness. So far, this kind of result has been given only for the case when the surfaces surrounding the fluid are fixed ($v_0 = 0$), corresponding for example to the injection of a fluid through a fixed gap. Whether the value of v_0 is zero or not, the system consists of two equations : a generalized Buckley-Leverett equation and a generalized Reynolds lubrication equation.

However, the assumption of zero value for v_0 is not realistic for most of the lubrication problems in which the fluid is sheared, due to the difference of velocities between the surrounding surfaces. Thus, it is the purpose of this chapter to give an existence and uniqueness result for the asymptotic system with non zero value for the shear velocity. The main difficulty comes from the study of the generalized Buckley-Leverett equation. More precisely, taking the fact that v_0 is different from 0 prevents us from using the classical results about first order hyperbolic equations: in fact, the flux function is not autonomous and we have to guarantee that the saturation lies in the interval $[0, 1]$, although the maximum principle is not a priori guaranteed anymore.

This chapter is organized as follows:

- Section 4.1 deals with the governing equations of the asymptotic system, obtained from the multifluid Stokes system. Thus, the generalized Buckley-Leverett / Reynolds system is presented, along with the physical assumptions related to realistic modellings.

- Section 4.2 is devoted to the analysis of the generalized Buckley-Leverett equation. Thus, we present the definition of a weak entropy solution of a scalar conservation law on a bounded domain, and we give some stability results (in particular, the saturation is a function with values in $[0, 1]$), along with an existence and uniqueness theorem, by using the concept of "semi Kružkov entropy-flux pairs". Moreover, we use a numerical scheme that allows to approximate the unique (physical) solution as the mesh size tends to 0.

- In Section 4.3, we present the analysis of the generalized Reynolds equation. In particular, we state an existence and uniqueness result for the pressure, and also a priori estimates in the H^1 or L^∞ norm which do not depend on the ratio ε of the viscosities.

- Section 4.4 deals with numerical computations. In particular, we present some numerical tests showing the importance of the shear effects on the saturation, pressure and velocity profiles but also on the boundary conditions. Then, we focus on cavitation phenomena : we show how our approach allows us to give some comprehensive details on the way the bifluid behaves in thin films. In particular, we compare the solution of the generalized Buckley-Leverett / Reynolds model to the solution of the Elrod-Adams model, which is a frequently used model in mechanical studies. This states that the Elrod-Adams model (which is heuristic) may be justified, at least numerically, by the bifluid model with an appropriate set of data.

4.1 Governing equations

We first recall the set of equations derived by Paoli in [Pao03]. Let be $\Omega =]0, L[$ and let us denote by $\partial\Omega = \{0, L\}$ its boundary, by Q_T the set $]0, T[\times\Omega$ and by Σ_T the set $]0, T[\times\partial\Omega$. We introduce the ratio $\varepsilon = \mu_g/\mu_l$, μ_l (resp. μ_g) being the viscosity of a fluid in liquid (resp. gaseous) phase. In view of cavitation-related phenomena, the fluid is supposed to be a lubricant: thus, $\mu_l \equiv \mu$, the liquid phase lubricant being considered as a reference fluid, and the gaseous phase lubricant may be considered as air or gas. In that configuration, typically $\varepsilon \sim 10^{-3}$. Now we introduce the main equations:

- Generalized Buckley-Leverett equation: the saturation s is governed by a scalar conservation law:

$$\frac{\partial}{\partial t}(h(x)s(t,x)) + \frac{\partial}{\partial x}(Q_{in}(t)f(s) + v_0 h(x)g(s)) = 0, \quad (t,x) \in Q_T, \tag{4.1}$$

  where h is the normalized gap between the surfaces, Q_{in} is the flow input, v_0 is the shear velocity corresponding to the speed of the lower surface, and s denotes the reference (liquid) fluid saturation. The functions f and g, defining the flux, are described later. However, we point out the fact that f represents the classical contribution to the Buckley-Leverett flux, while g is a non-classical contribution induced by shear effects. Equation (4.1) is completed with the following initial and boundary conditions

$$s(0,\cdot) = s_0, \quad \text{on } \Omega \tag{4.2}$$

$$\text{“}s = s_1\text{”}, \quad \text{on } \Sigma_T \tag{4.3}$$

where the sense of the boundary condition (4.3) has to be precised, as it will be discussed later.

This initial boundary value problem is weakly coupled with the following problem:

- Generalized Reynolds equation: for a given saturation s, the pressure p obeys the following law:

$$\frac{\partial}{\partial x}\left(A(s)\frac{h^3}{6\mu_l}\frac{\partial p}{\partial x}\right) = v_0\frac{\partial}{\partial x}\left(B(s)h\right), \quad (t,x) \in Q_T, \tag{4.4}$$

with the boundary condition:

$$p = 0, \quad (t,x) \in \Sigma_T. \tag{4.5}$$

Let us mention that expressions of functions f, g, A and B are fully detailed in the Appendix (at the end of this chapter). In [Pao03], the derivation of the coupled problem has been done under the assumption that the free boundary, which separates both phases, is a function belonging to $L^\infty((0,T); BV(\Omega))$. Thus, this assumption (on the shape and on the regularity of the free boundary) prevents us from considering multi-layer flows, although they are supposed to be relevant (see, for instance, [Pao03]). The main reason for restricting ourselves to this particular type of free boundaries lies in the difficulty to compute an explicit expression of f and g otherwise. Therefore, we will restrict ourselves to cases (i) and (ii) (in which f, g, A and B can be fully computed):

(i) The reference (liquid) fluid is adhering to the lower (moving) surface (see FIG.4.1).

(ii) The reference (liquid) fluid is adhering to the upper (fixed) surface (see FIG.4.2).

We will discuss in Section 4.4 whether the choice of each assumption is relevant or not. Notice that f, g, A and B highly depend on the ratio ε. As it will be pointed out further, the shape of the flux functions and coefficients remains the same for both cases, as described by the forthcoming assumptions.

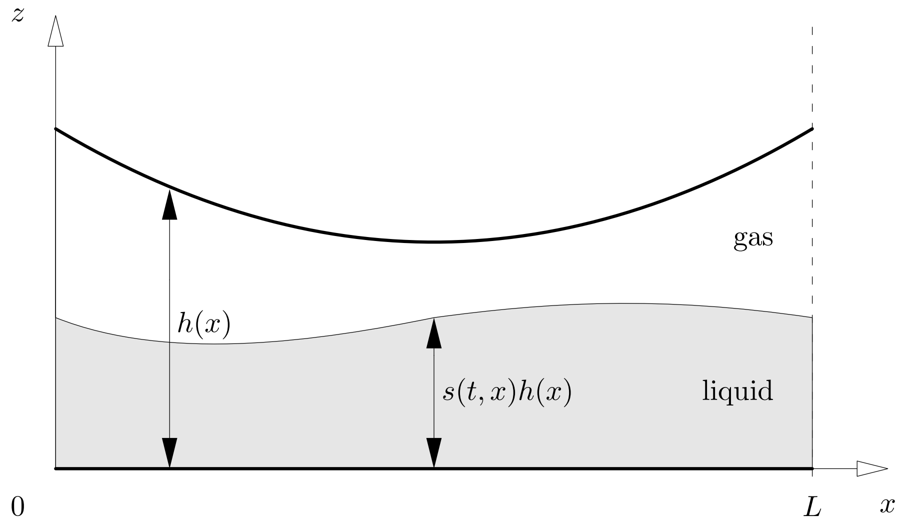


Figure 4.1. Case (i). The liquid phase is adhering to the lower surface

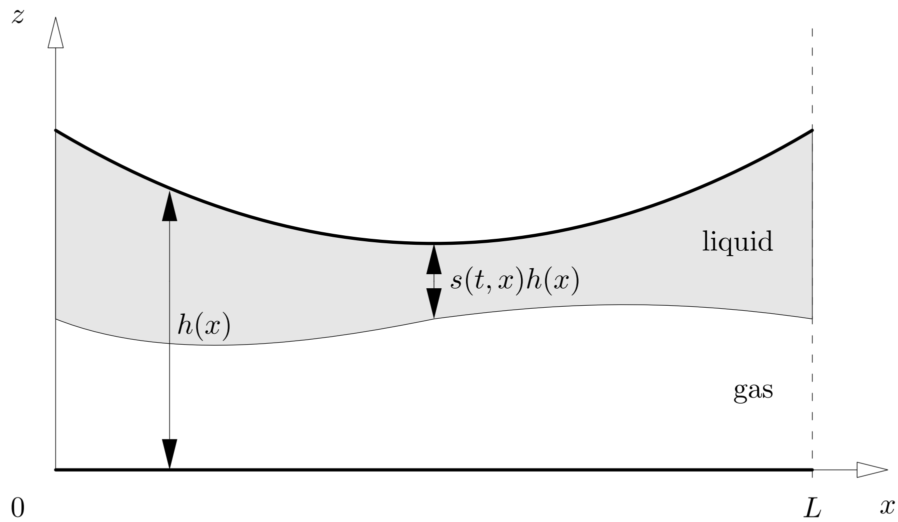


Figure 4.2. Case (ii). The liquid phase is adhering to the upper surface

We consider the following assumptions on the data:

Assumption 15 (initial and boundary functions)**.**

(i) $s_0 \in L^\infty(\Omega; [0,1])$,

(ii) $s_1 \in L^\infty(\mathbb{R}^+; [0,1])$.

Assumption 16 (flow, shear velocity)**.**

(i) $Q_{in} \in C^0([0,+\infty])$,

(ii) $\exists\ Q_{\min}, Q_{\max},\ 0 < Q_{\min} \leq Q_{in} \leq Q_{\max}$,

(iii) $v_0 > 0$.

Assumption 17 (gap between the surfaces)**.**

(i) $h \in C^1(\mathbb{R})$,

(ii) $\exists\ h_{\min}, h_{\max},\ 0 < h_{\min} \leq h \leq h_{\max}$.

Assumption 18 (auxiliary flux functions)**.**

(i) $f \in C^2([0,1])$, $f(0) = 0$, $f(1) = 1$, f *is a non-decreasing function,*

(ii) $g \in C^2([0,1])$, $g(0) = g(1) = 0$,

Our purpose is to state an existence and uniqueness result for problem (4.1)–(4.5). In fact, the main difficulty is to state an existence and uniqueness result for the generalized Buckley-Leverett equation (4.1)–(4.3). Moreover, since s denotes a saturation, we have also to state that the (possible) solution takes its values in $[0,1]$. Indeed, let us recall that the derivation of the generalized Buckley-Leverett equation is not fully rigorous in the sense that a strong assumption on the free boundary shape has been used. Let us recall also that the study of the generalized Buckley-Leverett equation including the shear term has been omitted by Paoli in [Pao03].

4.2 The generalized Buckley-Leverett equation

In a first subsection, we introduce an auxiliary problem and a corresponding "weak entropy solution", whose framework lies in the theory of scalar conservation laws on bounded domains. After establishing some results (existence, uniqueness, stability) on the properties of the auxiliary problem, we will prove, in a second subsection, how it is possible to reduce the generalized Buckley-Leverett problem to the auxiliary one. Finally, in the third subsection, we propose a numerical scheme whose solution converges to the "physical" solution (i.e. the weak entropy solution).

4.2.1 An auxiliary problem

Let us consider the following assumption:

Assumption 19. *The "auxiliary boundary / initial data" satisfy :*

(i) $u_0 \in L^\infty(\Omega; [0, 1])$,

(ii) $u_1 \in L^\infty(\mathbb{R}^+; [0, 1])$.

The "auxiliary gap" satisfies:

(i) $k \in C^1(\mathbb{R}^2)$,

(ii) $\exists\ k_{\min}, k_{\max},\ 0 < k_{\min} \leq k \leq k_{\max}$.

We introduce the following scalar conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(f(u) + k(t,x)\ g(u)\right) = 0, \quad (t,x) \in]0, \widetilde{T}[\times]0,1[\tag{4.6}$$

where f, g have been already defined and $\widetilde{T} > 0$. Equation (4.6) is completed with the following initial and boundary conditions

$$u(0,\cdot) = u_0, \qquad \text{on }]0,1[, \tag{4.7}$$

$$\text{“}u = u_1\text{”}, \qquad \text{on }]0, \widetilde{T}[\times\partial]0,1[. \tag{4.8}$$

Existence and uniqueness of a solution for scalar conservation laws on unbounded domains has been solved in the pioneering work of Kružkov [Kru70] who introduced the concept of weak entropy solution and related "Kružkov entropy-flux pairs". When dealing with bounded domains, introducing boundary conditions must be understood in a particular way: in the bounded domain setting, Bardos, Le Roux and Nédélec [BLRN79] also proved existence and uniqueness of a weak entropy solution satisfying a "Kružkov entropy-flux pair" formulation which includes boundary terms, under some regularity assumptions on the data. In particular, the way the boundary condition is satisfied is known as the "BLN" condition. Nevertheless, this formulation would not be sufficient by two reasons:

1) boundary and initial conditions lack regularity in comparison to the framework of [BLRN79],

2) it does not provide a stability result (we recall that we have to state that the possible solution is a function with values in $[0, 1]$).

The notion of weak entropy solution in the L^∞ framework is essentially due to Otto [Ott96] who introduced the so-called "boundary entropy-flux pairs". A more complete exposition appears in [MNRR96]. In fact, as it will be pointed out further, it is equivalent to use the "semi Kružkov entropy-flux pairs" and the "boundary entropy-flux pairs", at least in the case of a scalar conservation law with an autonomous flux. In the case of scalar conservation laws with non-autonomous fluxes, a "boundary entropy-flux pairs" formulation is not obvious anymore, but using the concept of "semi Kružkov entropy-flux pairs" [Car99, Ser96, Vov02] allows us to generalize the notion of entropy solution, state a stability result and prove existence and uniqueness of such a solution. Let us first introduce the definition of a weak entropy solution for the auxiliary problem (4.6)–(4.8) and state the existence and uniqueness result as a theorem.

Definition 9. *Let us suppose that Assumption 19 holds. A function $u \in L^\infty(Q_T)$ is said to be a weak entropy solution of problem (4.6)–(4.8) if it satisfies the inequality*

$$\begin{aligned}\int_0^1 \int_0^{\widetilde{T}} \Big\{ (u-\kappa)^\pm \, \frac{\partial \varphi}{\partial t} \;&+\; \Big(\Phi^\pm_{[f]}(u,\kappa) + k\ \Phi^\pm_{[g]}(u,\kappa)\Big)\ \frac{\partial \varphi}{\partial x} \\ &-\; \mathrm{sgn}_\pm(u-\kappa)\ \frac{\partial k}{\partial x}\ g(\kappa)\ \varphi \Big\}\ dx\ dt \\ &+\; \int_\Omega (u_0(x)-\kappa)^\pm \varphi(0,x)\ dx \\ &+\; \mathcal{M} \int_0^{\widetilde{T}} (u_1(t,1)-\kappa)^\pm \varphi(t,1)\ dt \\ &+\; \mathcal{M} \int_0^{\widetilde{T}} (u_1(t,0)-\kappa)^\pm \varphi(t,0)\ dt \geq 0, \end{aligned} \tag{4.9}$$

for all $\kappa \in [0,1]$, $\varphi \in \mathcal{D}([0,\widetilde{T}[\times[0,1])$ and $\varphi \geq 0$, the constant $\mathcal{M}$ being defined by

$$\mathcal{M} = \|f'\|_{L^\infty(0,1)} + k_{\max}\ \|g'\|_{L^\infty(0,1)}. \tag{4.10}$$

The functions $u \mapsto (u-\kappa)^\pm$ are the so-called "semi Kružkov entropies" (see [Car99, Ser96, Vov02]), defined by

$$(u-\kappa)^+ = \begin{cases} u-\kappa, & \text{if } u \geq \kappa, \\ 0, & \text{otherwise.} \end{cases} \qquad \text{and} \qquad (u-\kappa)^- = (\kappa-u)^+.$$

The functions $\Phi^\pm_{[f]}$ and $\Phi^\pm_{[g]}$ are the corresponding "semi Kružkov fluxes" defined by

$$\begin{aligned} \Phi^\pm_{[f]}(u,\kappa) &= \mathrm{sgn}_\pm(u-\kappa)(f(u)-f(\kappa)), \\ \Phi^\pm_{[g]}(u,\kappa) &= \mathrm{sgn}_\pm(u-\kappa)(g(u)-g(\kappa)), \end{aligned}$$

where $u \mapsto \mathrm{sgn}_{\pm}(u)$ *is the derivative of the function* $u \mapsto u^{\pm}$ *with value* 0 *at point* 0*.*

The functions $\Phi^{\pm}_{[f]}(\cdot,\kappa)$ and $\Phi^{\pm}_{[g]}(\cdot,\kappa)$ are defined on $[0,1]$ (as f and g). Therefore, both definitions of solutions (weak entropy and strong solutions) make sense only if the function u takes values in $[0,1]$ a.e., which is not supposed a priori in Definition 9. However, the set is preserved as shown by the following proposition.

Theorem 4.1 (Maximum principle)**.** *In Definition 9 and Equations (4.6)–(4.8), if we replace the functions* f*,* g*,* $\Phi^{\pm}_{[f]}$*,* $\Phi^{\pm}_{[g]}$ *by the functions* $\widetilde{f}$ *and* $\widetilde{g}$*,* $\widetilde{\Phi}^{\pm}_{[f]}$ *and* $\widetilde{\Phi}^{\pm}_{[g]}$ *defined by:*

$$\widetilde{f}(v) = \begin{cases} 0, & \text{if } v<0 \\ f(v), & \text{if } 0\le v\le 1 \\ 1, & \text{if } v>1 \end{cases}, \qquad \widetilde{\Phi}^{\pm}_{[f]}(v,\kappa) = \mathrm{sgn}_{\pm}(v-\kappa)(\widetilde{f}(v)-\widetilde{f}(\kappa)),$$

$$\widetilde{g}(v) = \begin{cases} 0, & \text{if } v<0 \\ g(v), & \text{if } 0\le v\le 1 \\ 0, & \text{if } v>1 \end{cases}, \qquad \widetilde{\Phi}^{\pm}_{[g]}(v,\kappa) = \mathrm{sgn}_{\pm}(v-\kappa)(\widetilde{g}(v)-\widetilde{g}(\kappa)),$$

and if u *is a weak entropy solution of problem (4.6)–(4.8), then* $0 \le u \le 1$ *a.e. on* $]0,\widetilde{T}[\times]0,1[$*.*

Proof. First let us notice that the following properties hold:

$$\begin{aligned} \widetilde{\Phi}^{\pm}_{[f]}(v,\kappa) &\le \|f'\|_{L^\infty(0,1)}(v-\kappa)^{\pm}, \\ \widetilde{\Phi}^{\pm}_{[g]}(v,\kappa) &\le \|g'\|_{L^\infty(0,1)}(v-\kappa)^{\pm}. \end{aligned}$$

Now, set $\kappa = 0$ in Inequality (4.9). Since we have $u_0^- = 0$, $u_1^- = 0$ (see Assumption 19), $g(0) = 0$ (see Assumption 18), the three last terms in (4.9) vanish. Thus, we have

$$\int_0^1 \int_0^{\widetilde{T}} \left(u^- \,\frac{\partial \varphi}{\partial t} + \left(\widetilde{\Phi}^{-}_{[f]}(u,0) + k\; \widetilde{\Phi}^{-}_{[g]}(u,0) \right) \frac{\partial \varphi}{\partial x} \right) dx\, dt \ge 0. \tag{4.11}$$

Now, let (τ, R) be such that $0<\tau<\widetilde{T}$, $\delta = \widetilde{T}-\tau$ and $0<R<1$. Let $r \in \mathcal{D}(\mathbb{R}^+)$ be such that: r is non-increasing, $r \equiv 1$ on $[0, R+\mathcal{M}\tau]$, $r \equiv 0$ on $[R+\mathcal{M}\tau+\delta/2, +\infty)$. Then, choosing

$$\varphi(t,x) = \frac{\tau - t}{\tau} \chi_{(0,\tau)}(t)\; r(x+\mathcal{M}t)$$

in Inequality (4.11) leads to

$$-\frac{1}{\tau}\int_0^1 \int_0^{\tau} u^- r(x+\mathcal{M}t)\, dt\, dx$$

$$+\int_0^1\int_0^\tau \frac{\tau-t}{\tau} r'(x+\mathcal{M}t)\ \Big(\mathcal{M}\ u^- + \widetilde{\Phi}^-_{[f]}(u,0) + k\ \widetilde{\Phi}^-_{[g]}(u,0)\Big) \geq 0. \tag{4.12}$$

Now, since $\widetilde{\Phi}^-_{[f]}(u,0) \leq \|f'\|_{L^\infty(0,1)} u^-$, $\widetilde{\Phi}^-_{[g]}(u,0) \leq \|g'\|_{L^\infty(0,1)} u^-$, we use Equality (4.10) to obtain:

$$\mathcal{M}\ u^- + \widetilde{\Phi}^-_{[f]}(u,0) + k\ \widetilde{\Phi}^-_{[g]}(u,0) \geq 0.$$

Moreover, since $r'(x+\mathcal{M}t) \leq 0$, the second term of the left-hand side of (4.12) is negative. Now, since $r(x+\mathcal{M}t) = 1$, for all $(t,x) \in (0,\tau)\times(0,R)$ and $r \geq 0$, the left-hand side of (4.12) is upper bounded by

$$-\frac{1}{\tau}\int_0^R\int_0^\tau u^-\,dx\,dt$$

which is consequently non-negative. Therefore, we have $u^- = 0$ on $(0,\tau)\times(0,R)$. Now, passing to the limit with $R \to 1$ and $\tau \to \widetilde{T}$, we have $u \geq 0$ a.e. Similarly, by choosing $\kappa = 1$ in Inequality (4.9) (with the semi entropies $u \mapsto (u-1)^+$), we prove that $u \leq 1$ a.e. on $]0,\widetilde{T}[\times]0,1[$. □

Remark 4.2. *Interestingly, the fact that the flux function in the auxiliary problem (4.6)–(4.8) is not autonomous involves a major difference with the autonomous case, concerning the stability intervals. Indeed, the following properties can be easily shown:*

- *if the flux is autonomous (for instance $k \equiv 0$) and if u_0, u_1 are functions with values in an interval $[a,b]$, then, u is a function with values in $[a,b]$;*
- *For the auxiliary problem, if u_0, u_1 are functions with values in an interval $[a,b] \subset [0,1]$, then, u is a function with values in $[0,1]$; thus, only the set $[0,1]$ is preserved. This is due to the properties of g, in particular $g(0) = g(1) = 0$.*

Remark 4.3. *A function u which satisfies Definition 9 is a weak solution in a classical sense. Indeed, for every $\varphi \in H^1_0(Q_T)$, we write $\varphi = \varphi^+ - \varphi^-$, with $\varphi^+ = \max(\varphi,0)$ and $\varphi^- = -\min(\varphi,0)$; obviously, $\varphi^\pm \in H^1_0(Q_T)$; thus adding the two inequalities (corresponding to each "semi Kružkov entropy-flux pair") gives:*

$$\int_{Q_T}\left\{\Big|u-\kappa\Big|\frac{\partial\varphi^\pm}{\partial t} + \operatorname{sgn}(u-\kappa)\Big((f(u)-f(\kappa)) + k\ (g(u)-g(\kappa))\Big)\ \frac{\partial\varphi\pm}{\partial x}\right.$$
$$\left. -\frac{\partial k}{\partial x}\ \operatorname{sgn}(u-\kappa)\ g(\kappa)\ \varphi^\pm\right\}\ dx\,dt \geq 0.$$

Now, taking $\kappa = \pm\|u\|_{L^\infty(Q_T)}$ gives:

$$\int_{Q_T}\left\{u\ \frac{\partial\varphi^\pm}{\partial t} + (f(u)+k\ g(u))\ \frac{\partial\varphi^\pm}{\partial x}\right\} dx\ dt = 0,$$

Now, by means of substraction, we immediatly obtain

$$\int_{Q_T} \left\{ u \; \frac{\partial \varphi}{\partial t} + (f(u) + k \; g(u)) \; \frac{\partial \varphi}{\partial x} \right\} dx \; dt = 0,$$

for every $\varphi \in H_0^1(Q_T)$, so that Equation (4.6) is gained in a weak sense.

Let us explain the way the boundary condition is satisfied:

Definition 10 (Boundary entropy-flux pairs, [Ott96])**.**
Let us define the flux $\hat{f} : (t,x,s) \longmapsto f(s) + k(t,x) \; g(s)$. The pair (H,Q) belonging to $C^1(\mathbb{R}^2) \times C^1(]0,\widetilde{T}[\times]0,1[\times\mathbb{R}^2)$ is said to be a "boundary entropy-flux pair" (for the flux $\hat{f}$) if it satisfies:

1. *for all $w \in \mathbb{R}$, $s \mapsto H(s,w)$ is a convex function,*
2. $\forall (t,x) \in]0,\widetilde{T}[\times]0,1[$, $\forall w \in \mathbb{R}$, $\partial_3 Q(t,x,s,w) = \partial_1 H(s,w) \partial_3 \hat{f}(t,x,s)$,
3. $\forall w \in \mathbb{R}$, $H(w,w) = 0$, $Q(\cdot,\cdot,w,w) = 0$, $\partial_1 H(w,w) = 0$.

Proposition 4.4 (Boundary condition, [Ott96])**.**
Let $u \in L^\infty(]0,\widetilde{T}[\times]0,1[)$ be a weak entropy solution of problem (4.6)–(4.8). Then,

$$\operatorname{ess}\lim_{\varrho \to 0^+} \int_0^{\widetilde{T}} \Big(Q(t,1,u(t,1-\varrho),u_1(1)) \; \beta(t,1) - Q(t,0,u(t,\varrho),u_1(0)) \; \beta(t,0) \Big) \; dt \geq 0,$$

(4.13) *for all "boundary entropy-flux pairs" (H,Q), $\forall \beta \in L^1(\Sigma_T)$, $\beta \geq 0$ a.e.*

Remark 4.5. *Now let us give some comprehensive details on the way to understand the boundary condition. This has been given in [MNRR96, Ott96, Vov02]: in general, the problem should be overdetermined and the boundary equality cannot be required to be assumed at each point of the boundary, even if the solution is a regular function. But, with additional assumptions, the more comprehensive "BLN" condition is recovered:*

(*i*) *If u admits a trace, i.e. there exists $u_{|b} \in L^\infty(]0,\widetilde{T}[\times\partial]0,1[)$ such that*

$$\operatorname{ess}\lim_{\varrho \to 0^+} \int_0^{\widetilde{T}} |u(t,1-\varrho) - u_{|b}(t,1)| + |u(t,\varrho) - u_{|b}(t,0)| \; dt = 0,$$

then, Inequality (4.13) is equivalent to the following inequality (see [DL88, Ott96])

$$Q(\cdot,1,u_{|b}(\cdot,1),u_1(\cdot,1)) \geq 0, \quad \text{a.e. on }]0,\widetilde{T}[, \tag{4.14}$$

$$Q(\cdot,0,u_{|b}(\cdot,0),u_1(\cdot,0)) \leq 0, \quad \text{a.e. on }]0,\widetilde{T}[. \tag{4.15}$$

Moreover, considering the particular "boundary fluxes"

$$H_\delta^+(z,w) = \Big((\max(z-w,0))^2+\delta^2\Big)^{1/2}-\delta, \tag{4.16}$$

$$Q_\delta^+(t,x,z,w) = \int_w^z \partial_1 H_\delta^+(\lambda,k)\ \partial_3\hat{f}(t,x,\lambda)\ d\lambda, \tag{4.17}$$

and

$$H_\delta^-(z,w) = \Big((\min(w-z,0))^2+\delta^2\Big)^{1/2}-\delta, \tag{4.18}$$

$$Q_\delta^-(t,x,z,w) = \int_w^z \partial_1 H_\delta^-(\lambda,k)\ \partial_3\hat{f}(t,x,\lambda)\ d\lambda, \tag{4.19}$$

and letting $\delta \to 0$, we have the following uniform convergences:

$$Q_\delta^\pm(t,x,s,w) \to \mathrm{sgn}_\pm(s-w)\left(\hat{f}(t,x,s)-\hat{f}(t,x,w)\right).$$

Finally, taking the following "boundary flux" in Inequalities (4.14) and (4.15)

$$\begin{aligned} Q(t,x,s,w) \quad = \quad & \mathrm{sgn}_+(s-\max(w,k))\left\{\hat{f}(t,x,s)-\hat{f}(t,x,\max(w,k)))\right\} \\ & +\mathrm{sgn}_-(s-\min(w,k))\left\{\hat{f}(t,x,s)-\hat{f}(t,x,\min(w,k)))\right\} \end{aligned}$$

yields to the classical BLN condition given by Bardos, Le Roux and Nédélec [BLRN79], that is:

$$\mathrm{sgn}(u_{|b}(t,1)-u_1(t,1))(\hat{f}(t,1,u_{|b}(t,1))-\hat{f}(t,1,k)) \geq 0, \tag{4.20}$$

$$\mathrm{sgn}(u_{|b}(t,0)-u_1(t,0))(\hat{f}(t,0,u_{|b}(t,0))-\hat{f}(t,0,k)) \leq 0, \tag{4.21}$$

for a.e. $(t,r) \in]0,\widetilde{T}[\times\partial]0,1[$, $\forall k \in [\min(u_{|b},u_1),\max(u_{|b},u_1)]$.

(ii) *Assume that u admits a trace on the boundary, then Inequalities (4.20) and (4.21) can be simplified in the following cases:*

- *If $\partial_3\hat{f}$ is a positive function on $]0,1[$, then $u(\cdot,0)=u_1$ (and nothing is imposed at $x=1$, i.e. the condition $u(\cdot,1)=u_1$ is not active).*
- *If $\partial_3\hat{f}$ is a negative function on $]0,1[$, then $u(\cdot,1)=u_1$ (and nothing is imposed at $x=0$, i.e. the condition $u(\cdot,0)=u_1$ is not active).*

Thus the boundary conditions may be "active" only on a part of the boundary.

Unfortunately, in the case of the flux $\hat{f}$ with Assumptions 18 and 19, monotonicity with respect to the third variable lacks, and we have to deal with "relaxed" Dirichlet boundary conditions.

Now, we conclude this subsection with the main result related to the auxiliary problem:

Theorem 4.6. *Under Assumption 19, problem (4.6)–(4.8) admits a unique weak entropy solution.*

Proof. Existence is proved using the (classical) parabolic approximation, which consists of adding an artificial diffusive term in the right-hand side of the hyperbolic equation (vanishing viscosity method). Next, passing to the limit on the diffusive parameter gives the existence result. Uniqueness is obtained from the Kružkov method of doubling variables. The complete proof is even valid for first order quasilinear equations, and completely detailed in Chapter 3. □

4.2.2 Existence and uniqueness of a weak entropy solution for the Buckley-Leverett problem

In this subsection, we show that problem (4.1)–(4.3) can be reduced to an auxiliary problem as the one described in the previous subsection, namely problem (4.6)–(4.8). Then, it suffices to use the results established for the auxiliary problem.

Definition 11 (Direct reduction)**.** *Let us consider the following changes of variables:*

$$Y(x) = L\,\frac{\displaystyle\int_0^x h(t)\ dt}{\displaystyle\int_0^L h(x)\ dx}, \qquad \mathcal{T}(t) = L\,\frac{\displaystyle\int_0^t Q_{in}(s)\ ds}{\displaystyle\int_0^L h(x)\ dx}.$$

We also define the inverse functions of Y and $\mathcal{T}$, respectively denoted Y^{-1} and $\mathcal{T}^{-1}$.

Definition 12 (Inverse reduction)**.** *Let Y^{-1} and $\mathcal{T}^{-1}$ be defined as the respective unique solution of the following Cauchy problems:*

$$\begin{cases} \dfrac{dY^{-1}}{dy}(y) &=& \dfrac{1}{L\ h(Y^{-1}(y))}\displaystyle\int_0^L h(x)\ dx, \\ Y^{-1}(0) &=& 0, \end{cases}$$
$$\begin{cases} \dfrac{d\mathcal{T}^{-1}}{d\tau}(\tau) &=& \dfrac{1}{L\ Q_{in}\left(\mathcal{T}^{-1}(\tau)\right)}\displaystyle\int_0^L h(x)\ dx, \\ \mathcal{T}^{-1}(0) &=& 0. \end{cases}$$

Remark 4.7. *Y is an increasing function which defines an isomorphism from $[0, L]$ to $[0, 1]$. In the same way, $\mathcal{T}$ is an increasing function which defines an isomorphism from $[0, T]$ to $[0, \widetilde{T}]$, with*

$$\widetilde{T} = L \, \frac{\displaystyle\int_0^T Q_{in}(s)\ ds}{\displaystyle\int_0^L h(x)\ dx}.$$

Proposition 4.8. *Problem (4.1)–(4.3) can be reduced to an auxiliary problem (4.6)–(4.8).*

Proof. First, we define

$$u(\mathcal{T}(t), Y(x)) = s(t, x), \tag{4.22}$$

so that Equation (4.1) reduces to

$$\frac{\partial}{\partial \tau} u(\tau, y) + \frac{\partial}{\partial y} \left(f(u) + k(\tau, y)\ g(u) \right) = 0, \quad (\tau, y) \in (0, \widetilde{T}) \times (0, 1) \tag{4.23}$$

with $k(\tau, y) = \dfrac{v_0}{\left(Q_{in} \circ \mathcal{T}^{-1}\right)(\tau)} \left(h \circ Y^{-1}\right)(y)$.

The initial and boundary conditions are modified as follows:

$$u(0, y) = u_0(0, y) = s_0 \circ Y^{-1}(y), \qquad y \in]0, 1[\tag{4.24}$$

$$\text{“}u(\tau, y) = u_1(\tau, y) = s_1\left(\mathcal{T}^{-1}(\tau), Y^{-1}(y)\right)\text{”}, \quad (\tau, y) \in (0, \widetilde{T}) \times \partial]0, 1[\tag{4.25}$$

□

As a consequence, turning back to the original variables immediatly gives:

Definition 13. *A function $s \in L^\infty(Q_T; [0, 1])$ is said to be a weak entropy solution of problem (4.1)–(4.3) if it satisfies*

$$\begin{aligned}
\int_{Q_T} \Bigg\{ h(x)\ (s - \kappa)^{\pm}\ \frac{\partial \varphi}{\partial t} \quad &+ \quad \left(Q_{in}(t)\ \Phi^{\pm}_{[f]}(s, \kappa) + v_0 h(x)\ \Phi^{\pm}_{[g]}(s, \kappa) \right) \frac{\partial \varphi}{\partial x} \\
&- \quad v_0\ \mathrm{sgn}_{\pm}(s - \kappa)\ h'(x)\ g(\kappa)\ \varphi \Bigg\} \, dx\, dt \\
&+ \quad \int_\Omega h(x)\ (s_0(x) - \kappa)^{\pm} \varphi(0, x)\, dx \\
&+ \quad \mathcal{L} \int_0^T (s_1(t, 1) - \kappa)^{\pm} \varphi(t, 1)\, dt \\
&+ \quad \mathcal{L} \int_0^T (s_1(t, 0) - \kappa)^{\pm} \varphi(t, 0)\, dt \geq 0
\end{aligned} \tag{4.26}$$

for all $\kappa \in [0,1]$, $\varphi \in \mathcal{D}([0,T[\times\mathbb{R})$, $\varphi \geq 0$. *Here, we can choose:*

$$\mathcal{L} = Q_{\max} \max(\|f'\|_{L^\infty(0,1)}) + v_0 h_{\max} \frac{Q_{\max}}{Q_{\min}} \max(\|g'\|_{L^\infty(0,1)}).$$

Theorem 4.9. *Under Assumptions 15–18, problem (4.1)–(4.3) admits a unique weak entropy solution.*

4.2.3 Numerical scheme for the generalized Buckley-Leverett equation

We study the numerical method simulating the solution of problem (4.23)–(4.25) (for convenience) which is equivalent to problem (4.1)–(4.3) up to a change of variables.

The following theorem gives a convergence result which enables us to use some classical finite volume methods (such as the Lax-Friedrichs scheme, for instance) in order to compute the weak entropy solution.

Theorem 4.10 (Vovelle [Vov02]). *Let us consider a finite volume scheme with monotone fluxes associated to problem (4.23)–(4.25) and its corresponding numerical solution* $u_{\mathcal{T},k}$. *Then* $(u_{\mathcal{T},k})$ *strongly converges to the weak entropy solution* u *in* $L^p_{loc}(\mathbb{R}^+, \Omega)$ *for every* $p \in [2, +\infty[$.

From a theoretical point of view, it appears that the simulation of the Buckley-Leverett problem can be easily done thanks to the earlier convergence result. Nevertheless, let us recall that f and g highly depend on the ratio of the viscosities, namely ε (see FIG.4.14, 4.15 and 4.16 in the appendix of the chapter). In practical situations, ε may be small (10^{-3} for an air-liquid mixture). Thus, let us study the behaviour of the scheme for small values of ε. For this, let us consider a uniform admissible mesh, whose step size is denoted Δx, with a time step Δt. We first notice that:

- $\|f'\|_{L^\infty}$ and $\|g'\|_{L^\infty}$ tend to explode as ε tends to 0. More precisely, we have (after omitted computations)

$$\|f'\|_{L^\infty} = \mathcal{O}\left(\varepsilon^{-1/3}\right), \quad \|g'\|_{L^\infty} = \mathcal{O}\left(\varepsilon^{-1/3}\right),$$

  so that we get also $\mathcal{M} = \mathcal{O}\left(\varepsilon^{-1/3}\right)$ (see Equation (4.10)).

- Moreover, Id denoting the identity application, we have that

$$\|f\|_{L^1} = \mathcal{O}\left(\varepsilon^{1/3}\right), \quad \|\mathrm{Id} - g\|_{L^1} = \mathcal{O}\left(\varepsilon^{1/3}\right).$$

Then, Δx needs to be adapted to the value of ε in order to describe phenomena related to the boundary layer, so that the mesh size should satisfy

$$\Delta x = \mathcal{O}(\varepsilon^{1/3}) \tag{4.27}$$

in order to describe boundary layers of f and g.

Now, we recall that the following CFL condition has to be imposed in order to ensure the stability of the numerical scheme:

$$\exists \xi \in]0,1[, \ \Delta t \leq (1-\xi)\frac{\Delta x}{\mathcal{M}}.$$

Thus, we obtain the order of the time step:

$$\Delta t = \mathcal{O}(\varepsilon^{2/3}). \tag{4.28}$$

Now, it clearly appears that it becomes difficult to simulate the Buckley-Leverett equation for too small values of ε. Indeed, for each time step, the number of computations increases with $1/\Delta x$; similarly, the number of times steps increases with $1/\Delta t$ so that it becomes more and more difficult to attain a (possible) stationary solution by passing to the limit in the evolutive problem.

4.3 The generalized Reynolds equation

4.3.1 Existence and uniqueness

Definition 14. *Let $s \in L^\infty(Q_T;[0,1])$. A function $p \in L^\infty((0,T);H_0^1(\Omega))$ is a weak solution of problem (4.4)–(4.5) if it satisfies*

$$\int_0^L A(s)\frac{h^3}{6\mu_l}\frac{\partial p}{\partial x}\frac{\partial v}{\partial x}\,dx = v_0\int_0^L B(s)h\,\frac{\partial v}{\partial x}\,dx, \quad \forall v \in H_0^1(\Omega) \tag{4.29}$$

for almost every $t \in (0,T)$.

Next, we establish the existence and uniqueness of solution for the generalized Reynolds equation:

Theorem 4.11. *Under Assumptions 15–18, problem (4.4)–(4.5) admits a unique solution p in the sense of Definition 14. Moreover, we have the following estimates:*

$$\left\|p\right\|_{L^\infty((0,T);H^1(\Omega))} \leq C_1, \qquad \left\|p\right\|_{L^\infty(Q_T)} \leq C_2,$$

uniformly with respect to ε*.*

Proof. Obviously, time t plays the role of a parameter. Thus, considering a fixed time t, existence and uniqueness of the pressure follows from Lax-Milgram lemma applied to an elliptic problem. To achieve the proof, it is sufficient to state that the coefficients are bounded and that the A is coercive, which is obvious from the following properties

$$\begin{array}{rlll} 1 \leq & A(s) \leq & 1/\varepsilon, & \forall s \in [0,1], \\ 0 \leq & B(s) \leq & 2, & \forall s \in [0,1], \end{array}$$

which are easily obtained from the definitions of A and B (see the Appendix at the end of the chapter). Here, ε denotes the ratio of the viscosities μ_g/μ_l. Thus, using $p(t,\cdot)$ ($t \in (0,T)$) as a test-function in the variational formulation (4.29), we have:

$$\begin{aligned} h_{\min}^3 \int_0^L \left|\frac{\partial p}{\partial x}(t,\cdot)\right|^2 \leq \int_0^L A(s(t,\cdot))\ h^3 \left|\frac{\partial p}{\partial x}(t,\cdot)\right|^2 &= 6v_0\mu_l \int_0^L B(s(t,\cdot))\ h\ \frac{\partial p}{\partial x}(t,\cdot) \\ &\leq 12v_0\mu_l\ h_{\max} \int_0^L \frac{\partial p}{\partial x}(t,\cdot). \end{aligned}$$

Next, using the Cauchy-Schwarz inequality, we have:

$$\left\|\frac{\partial p}{\partial x}(t,\cdot)\right\|_{L^2(\Omega)} \leq \frac{12v_0\mu_l h_{max} L^{1/2}}{h_{min}^3}.$$

The estimate in the H^1 norm is straightforward from Poincaré-Friedrichs inequality. The estimate in the L^∞ norm comes from the fact that $H^1(\Omega) \subset L^\infty(\Omega)$ with compact injection. Since the earlier inequality holds for almost every $t \in (0,T)$, the proof is concluded. □

4.3.2 Simulation of the generalized Reynolds equation

Suppose that the saturation s can be computed for each time step. Then, the pressure p is also obtained at each time step by using any numerical method related to an elliptic problem: finite difference discretization, shooting method with a Runge-Kutta solver (after some linear interpolation procedure on the saturation) or finite elements method.

4.4 Numerical simulation

4.4.1 Influence of the shear effects and boundary conditions

As pointed out in the previous sections, the shear effects play a crucial role on the analysis of the generalized Buckley-Leverett equation. Indeed, we have already mentioned that it leads to the non-autonomous property of the flux, and also to a relaxation of the Dirichlet conditions: without shear velocity, the boundary condition is active only at point $x = 0$ because of the monotonicity of f. But when the shear effects are included, this lacks due to the presence of the partial flux g. We are interested in the contribution of the shear term in the Buckley-Leverett equation. For this, we compare the following regimes:

- $v_0^{(1)} = 0$: injection of a fluid through a fixed gap,
- $v_0^{(2)} = 1$: injection of a fluid through a fixed gap between two surfaces in relative motion (shear effects).

The main difference lies in the properties of the Buckley-Leverett flux function. For this, we choose the following numerical data:

- The lubricant is adhering to the moving surface (Case (i)),
- Geometrical data: $\Omega =]0,1[$, $h(x) = (2x-1)^2 + \dfrac{1}{2}$,
- Reference viscosity: $\mu_l = \mu_g = 1$,
- Flow input: $Q_{in} = v_0^{(2)}\ \theta_{in} h(0)$ with $\theta_{in} = 0.37$,
- The mesh grid has 600 elements,
- The CFL condition is given by $\dfrac{\Delta t}{\Delta x}\mathcal{M} = 0.9$.

We start from two different initial conditions:

$$s_0^{(1)}(x) = (\theta_{in} - 0.01)(2x-1)^6 + 0.01, \quad s_0^{(2)}(x) = (\theta_{in} - 0.99)(2x-1)^6 + 0.99,$$

($s_0^{(i)}$ is called initial condition (i)) and we observe (see Fig.4.3–4.6) that in both regimes ($v_0 = 0$ or $v_0 \neq 0$), the numerical stationary solution (obtained for $T = T_f$) does not depend on the initial condition. Moreover, the shear effects involve a major difference with the autonomous case: the stationary solution is not constant but contains balanced effects due to the non-autonomous flux.

Now, let us focus on the boundary conditions. As it was pointed out, the boundary conditions may be active only on some part of the boundary. Moreover, Theorem 4.10

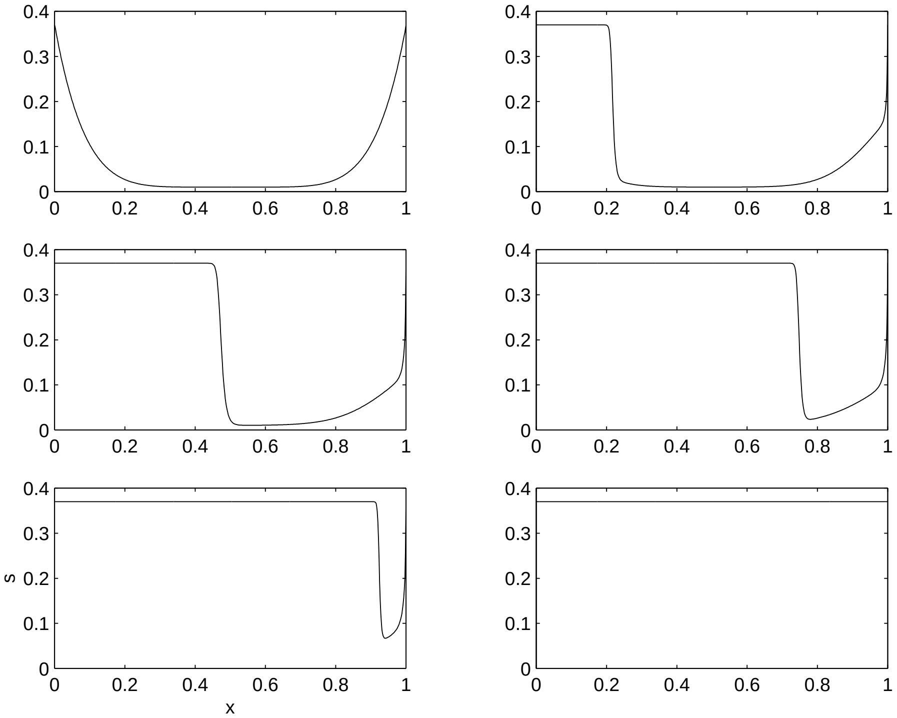


Figure 4.3. Saturation at different time steps $t = (i-1)T_f/5$ ($i = 1$ to 6, from left to right, top to bottom), without shear effects - Initial condition: $s_0^{(1)}$

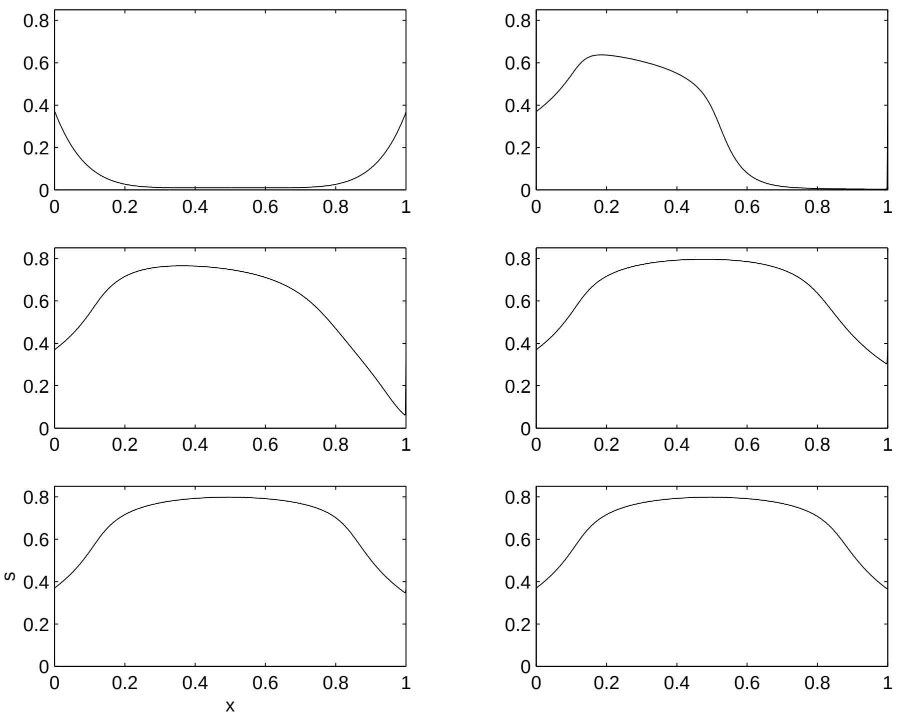


Figure 4.4. Saturation at different time steps $t = (i-1)T_f/5$ ($i = 1$ to 6, from left to right, top to bottom), including shear effects -Initial condition $s_0^{(1)}$

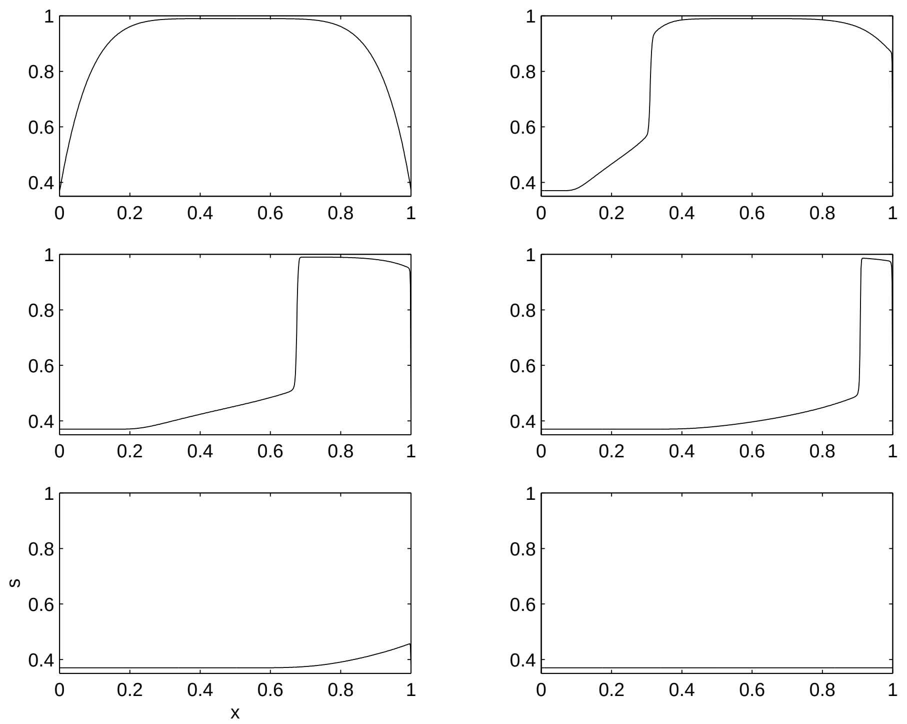


Figure 4.5. Saturation at different time steps $t = (i-1)T_f/5$ ($i = 1$ to 6, from left to right, top to bottom), without shear effects - Initial condition: $s_0^{(2)}$

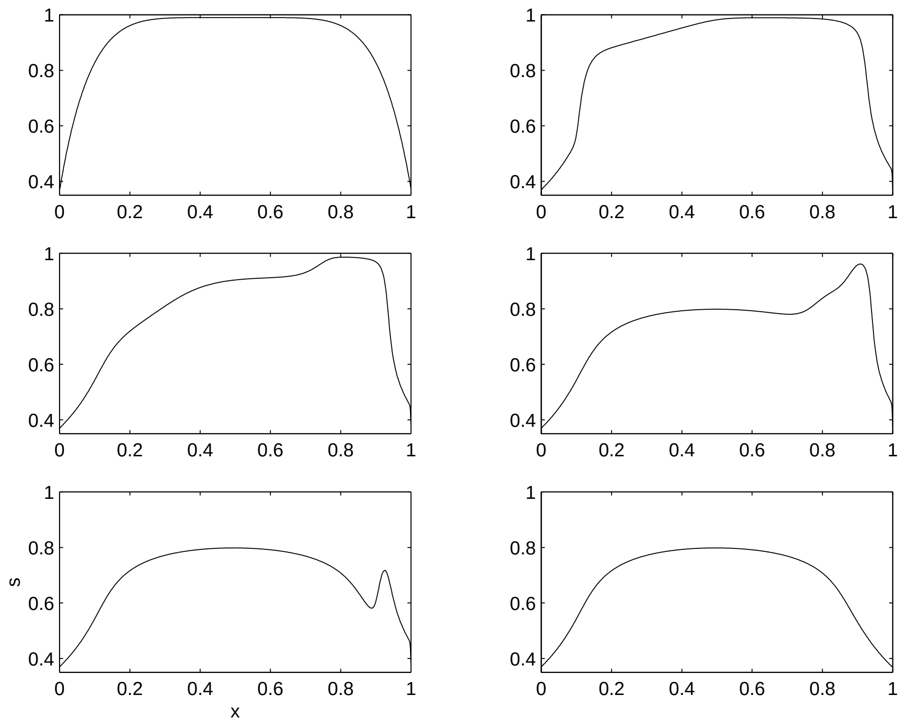


Figure 4.6. Saturation at different time steps $t = (i-1)T_f/5$ ($i = 1$ to 6, from left to right, top to bottom), including shear effects -Initial condition $s_0^{(2)}$

provides a strong convergence result, up to a possible boundary layer at each point of the boundary. This kind of behaviour is illustrated on FIG.4.7. As in the previous tests, the lubricant is adhering to the moving surface (Case (i)), the geometry is unchanged (converging-diverging profile) and the following numerical data have been considered:

- Shear velocity: $v_0 = 1$,
- Viscosities: $\mu_l = 1$, $\mu_g = 10^{-3}\mu_l$,
- Flow input: $Q_{in} = v_0\ h(0)\ g(\theta_{in})/(1 - f(\theta_{in}))$ with $\theta_{in} = 0.385$,
- The mesh grid has 1400 elements,
- The CFL condition is given by $\dfrac{\Delta t}{\Delta x}\mathcal{M} = 0.9$.

The initial condition is $s_0 \equiv \theta_{in}$ and the boundary condition also takes the value $s_1 = \theta_{in}$ (in the sense that has been precised before).

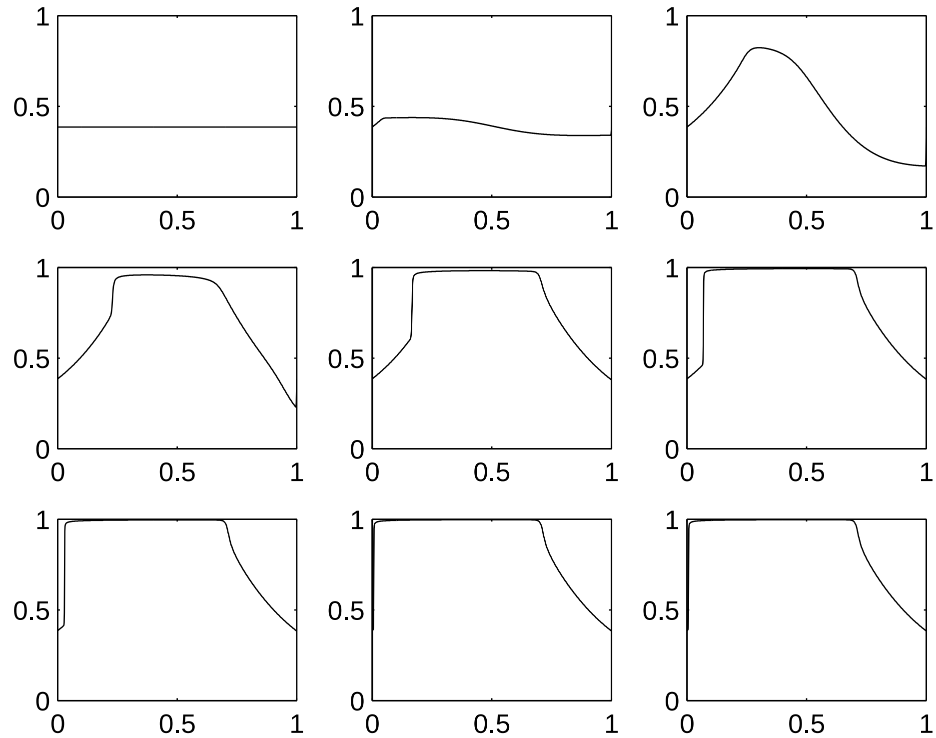


Figure 4.7. Saturation profile with partially active boundary conditions at different time steps: $t = (i-1)T_f/9$, for $i = 1$ to 9, from left to right, top to bottom

4.4.2 About cavitation phenomena in lubrication theory

Lubricated devices are generally made of two surfaces which are closely spaced, the annular gap being filled with some lubricant. The radial clearance is very small, typically $\Delta r/r = 10^{-3}$ for infinite oil lubricated bearings wich allows us to use the Reynolds equation:

$$\frac{d}{dx}\Big(\frac{h^3}{6\mu_l}\frac{dp}{dx}\Big) = v_0 \frac{dh}{dx}$$

where μ_l is the lubricant viscosity, p the pressure distribution, h the height between the two surfaces and v_0 the relative speed of the surfaces.

Nevertheless, this modelling does not take into account cavitation phenomena: cavitation is defined as the rupture of the continuous film due to the formation of air bubbles and makes the Reynolds equation no longer valid in the cavitation area. In order to make it possible, various models have been used, the most popular perhaps being variational inequalities which have a strong mathematical basis but lack physical evidence. Thus, a more realistic model, the Elrod-Adams model, is often used, assuming that the cavitation region is a fluid-air mixture and introducing an additional unknown θ (the saturation of fluid in the mixture) (see [CC83, CE70, CE71, EA75]). The model includes a modified Reynolds equation, with the following formulation:

$$(\mathcal{P}) \left\{ \begin{array}{l} \dfrac{d}{dx}\left(\dfrac{h^3}{6\mu_l}\ \dfrac{dp}{dx}\right) = v_0 \dfrac{d\theta h}{dx}, \\ p \geq 0, \quad 0 \leq \theta \leq 1, \quad p(1-\theta) = 0, \end{array} \right.$$

with, for instance, the boundary conditions:

- Dirichlet conditions : $p(L) = 0$,
- Neumann conditions : $v_0\theta(0)h(0) - \dfrac{h^3(0)}{6\mu_l}\ \dfrac{dp}{dx}(0) = Q_{in}$.

Introducing the domains

$$\begin{array}{lcll} \Omega^+ & = & \{x \in \Omega,\ p(x) > 0\} & \text{(lubricated region)}, \\ \Omega_0 & = & \{x \in \Omega,\ p(x) = 0\} & \text{(cavitated region)}, \\ \Sigma & = & \partial\Omega^+ \cup \Omega & \text{(free boundary)}, \end{array}$$

the free boundary Σ separates a full film area Ω^+ from a cavitated area Ω_0. Notice that Q_{in} denotes the flow input, which can be expressed as $Q_{in} = v_0\ \theta_{in} h(0)$, with

$$0 \leq \theta_{in} \leq 1.$$

This model has been studied in [BC83]. In particular, it appears that Neumann conditions are compatible with homogeneous Dirichlet conditions if and only if the normalized flow input θ_{in} belongs to some interval $]\theta_{\min}, \theta_{\max}[$. This corresponds to some "starvation" phenomenon: the domain is divided into three areas: starvation area ($\Omega_0^{(1)}$ in which $p \equiv 0$), then a satured area (Ω^+ in which $p > 0$) and then another cavitated area ($\Omega_0^{(2)}$ in which $p \equiv 0$).

Now, we want to compare the generalized Buckley-Leverett / Reynolds model to the Elrod-Adams model, which is motivated by the multifluid approach: indeed, the Elrod-Adams model describes partial lubrication of a device composed of a lubricant in two phases (liquid and gas) but strictly focuses on the liquid aspect. The generalized Buckley-Leverett / Reynolds model takes into account the influence of both phases. Thus, recalling that $\varepsilon \sim 10^{-3}$ in the case of a bifluid composed of liquid/gas, we want to numerically compare the two models. More precisely, we aim at comparing the numerical stationary solution of the generalized Buckley-Leverett / Reynolds problem to the solution of the Elrod-Adams problem. For this, let us motivate the choice of the data:

- Geometrical data: $\Omega =]0, 1[$, $h(x) = (2x-1)^2 + \dfrac{1}{2}$,
- Shear velocity, reference viscosity (lubricant viscosity) : $v_0 = 1$ and $\mu_l = 1$,
- Flow input : $\theta_{in} = 0.385$. Let us notice that the same flow input Q_{in} has been considered in the Buckley-Leverett / Reynolds model and the Elrod-Adams model.

The choice of such a flow input is designed to compare the Buckley-Leverett / Reynolds model to the Elrod-Adams model with the same data. With the chosen value, the Elrod-Adams solution contains starvation along with a well-known discontinuity in the saturation function corresponding to the rupture of the film thickness; thus, same phenomena observed with the Buckley-Leverett / Reynolds model would lead to some kind of a justification of the Elrod-Adams model. Unfortunately, it is impossible to obtain a Buckley-Leverett saturation with strictly value 1 at one point of the domain with such a choice, as it is stated in the following result:

Proposition 4.12. *If s is a stationary solution of the generalized Buckley-Leverett equations (4.1)–(4.3), then the following are equivalent:*

(i) There exists $x \in \Omega$ such that $s(x) = 1$.

(ii) The relative flow input satisfies the so-called "full saturation condition":

$$\theta_{in} = \frac{g(\theta_{in})}{1 - f(\theta_{in})}.$$

Proof. Let us assume that s is a stationary solution of Equation (4.1). Then, after integration, $Q_{in} f(s(x)) + v_0\ h(x)\ g(s(x)) = C$, $\forall x \in \Omega$, C being a constant. This constant is determined by the value of the earlier function at one boundary of Ω (notice that $h(0) = h(1)$), namely $C = Q_{in} f(\theta_{in}) + v_0\ h(0)\ g(\theta_{in})$. Since $f(1) = 1$ and $g(1) = 0$, if there exists $x_0 \in \Omega$, such that $s(x_0) = 1$, one gets

$$Q_{in} f(s(x_0)) + v_0\ h(x_0)\ g(s(x_0)) = Q_{in}. \tag{4.30}$$

Thus, from the previous equations,

$$Q_{in} = v_0\ h(0)\ \theta_{in} = v_0\ h(0)\ \frac{g(\theta_{in})}{1 - f(\theta_{in})}.$$

□

Proposition 4.13. *We have the following properties:*

- *For all $\theta_{in} \in]0, 1[$, if s is a stationary solution of the generalized Buckley-Leverett equations (4.1)–(4.3), then s does not reach the value 1.*
- *For all $\theta_{in} \in]0, 1[$, the "full saturation condition" is asymptotically satisfied, i.e.*

$$\lim_{\varepsilon \to 0} \frac{g(\theta_{in})}{1 - f(\theta_{in})} = \theta_{in}.$$

Proof.
• The proof is straightforward from the fact that

$$x > \frac{g(x)}{1 - f(x)}, \quad \forall x \in [0, 1[$$

so that the "full saturation condition" cannot be satisfied.
• Recalling the properties of f and g (see also FIG.4.14 and 4.15 in the Appendix), $\lim_{\varepsilon \to 0} f = 0$ and $\lim_{\varepsilon \to 0} g = \mathrm{Id}$, uniformly on every compact set $K \subset [0, 1[$. Thus, we have (for $\theta_{in} \neq 1$):

$$\lim_{\varepsilon \to 0} \left\{ Q_{in}\ f(\theta_{in}) + v_0\ h(0)\ g(\theta_{in}) \right\} = Q_{in},$$

which means that the "full saturation condition" is asymptotically satisfied. □

Remark 4.14. *From Propositions 4.12 and 4.13, it is impossible to get a (possible) stationary saturation with value 1 at any location of the domain, because the "full saturation condition" never holds (for non-pathological values of θ_{in}). Nevertheless, since it is asymptotically attained, we infer that considering small values of ε increases our hope to* *compare the Buckley-Leverett / Reynolds solution to the Elrod-Adams solution. Unfortunately, the behaviour of both fluxes f and g is pathological as ε tends to 0. Indeed, the graph of functions $x \mapsto f(x)$ and $x \mapsto g(x)$ clearly shows the presence of a boundary layer at $x = 1$ when ε tends to 0. Moreover, we have already mentioned that the numerical simulation of the Buckley-Leverett problem for small values of ε becomes difficult.*

Now let us enter into the details of the numerical comparison. Numerical data related to the simulation of the Buckley-Leverett equation are the following ones:

▷ The mesh grid has 1400 elements.

▷ The CFL condition is given by $\dfrac{\Delta t}{\Delta x}\mathcal{M} = 0.9$.

▷ The numerical stationary solution is attained for a relative error in the discrete L^2 norm with value 10^{-24}.

We study the behaviour of the bifluid in the two mentioned cases (see the Appendix at the end of the chapter).

Case (i): the liquid phase is adhering to the lower (moving) surface

Fig.4.8 (resp. 4.9) gives the pressure (resp. saturation) distribution obtained with the Buckley-Leverett / Reynolds model along with the solution of the Elrod-Adams model. We can see that when ε decreases, the Buckley-Leverett saturation tends to the Elrod-Adams saturation; nevertheless, it never reaches the value 1, as previously mentionned. For $\varepsilon = 10^{-3}$, which corresponds to physical situations in lubrication theory, the Buckley-Leverett saturation is very near to the Elrod-Adams one, up to the full saturation area. When considering the pressure, we can see that the peak of the Buckley-Leverett pressure never reaches the one of the Elrod-Adams pressure; this is due to the sensitivity to the coefficients in the generalized Reynolds equation. Indeed, the fact that the Buckley-Leverett saturation never reaches the value 1 prevents the generalized Reynolds pressure from approaching the Elrod-Adams pressure. Nevertheless, it is observed that non-positive pressures in the Buckley-Leverett / Reynolds model tend to vanish as ε tends to 0. Another interesting point is to consider that, using the Elrod-Adams saturation in the generalized Reynolds equation allows to exactly obtain the Elrod-Adams pressure, although the main differnece between the Elrod-Adams saturation and the Buckley-Leverett saturation lies in the (nearly) satured region (in the first case, the value 1 is reached although, in the second case, it never reaches this value); this observation shows how the "full saturation condition" is important. However, it is reasonable to say that computations for $\varepsilon = 10^{-3}$ allow to identify cavitated and satured areas.

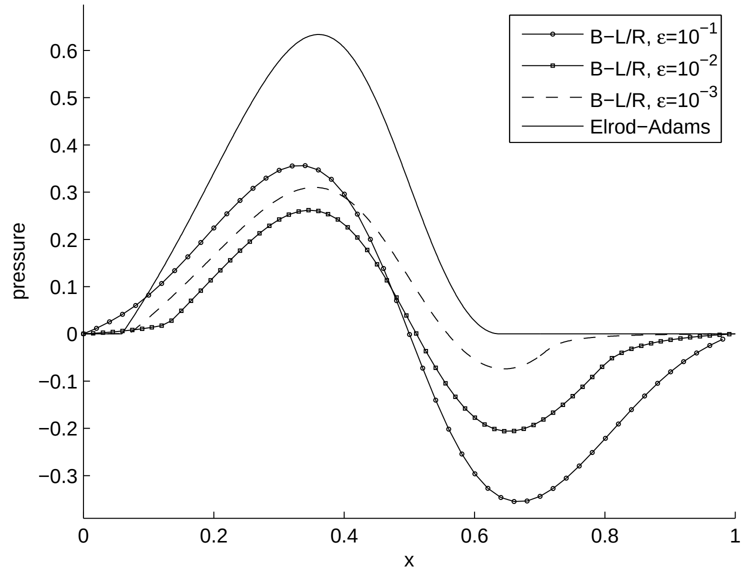


Figure 4.8. Pressure distribution for the Buckley-Leverett and Elrod-Adams models (Case (i))

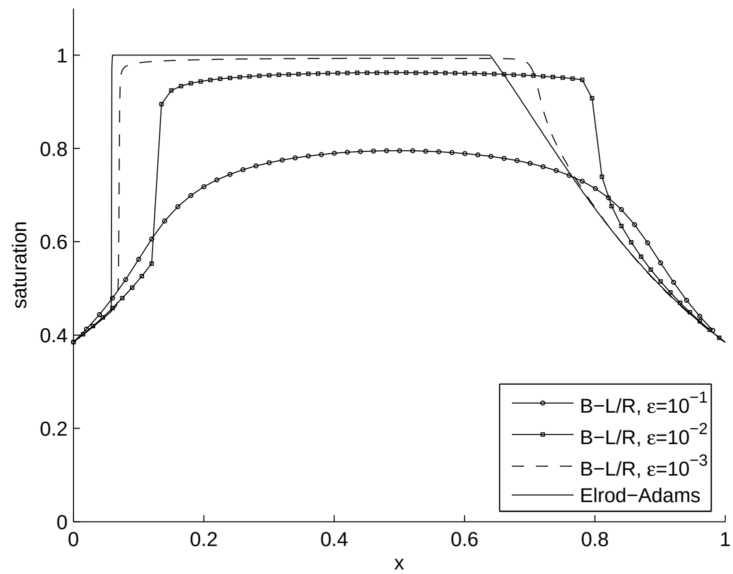


Figure 4.9. Saturation distribution for the Buckley-Leverett and Elrod-Adams models (Case (i))

FIG.4.10 describes the horizontal velocity distribution in the domain

$$\{(x,z) \in \mathbb{R}^2,\ 0 < x < 1,\ 0 < z < h(x)\}$$

with both models: the Buckley-Leverett / Reynolds model and the Elrod-Adams model.

- *Computation of the velocity for the Buckley-Leverett / Reynolds model:* Let us denote (U, V) the velocity in the bifluid. The following relationship has been derived in [Pao03]:

$$\eta(x,z)\,\frac{\partial^2 U}{\partial z^2}(x,z) = \frac{dp}{dx}(x), \tag{4.31}$$

$$U(x,0) = v_0, \tag{4.32}$$

$$U(x,h(x)) = 0, \tag{4.33}$$

  with

$$\eta(x,z) = \begin{cases} \mu_l, & \text{if } \ 0 < z < s(x)h(x), \\ \mu_g, & \text{if } \ s(x)h(x) < z < h(x). \end{cases}$$

  Moreover, $z \mapsto U(x,z)$ is continuous at the interface $z = s(x)h(x)$. Thus, integrating twice Equation (4.31) and taking account of boundary conditions (4.32) and (4.33) allow to get the velocity profile, up to the velocity at the free boundary, denoted $U^\star$. Nevertheless, $U^\star$ is obtained by taking account of

$$\lim_{z\to(s(x)h(x))^-}\left\{\mu_l\,\frac{\partial U}{\partial z}(x,z)\right\} = \lim_{z\to(s(x)h(x))^+}\left\{\mu_g\,\frac{\partial U}{\partial z}(x,z)\right\}.$$

- *Computation of the velocity for the Elrod-Adams model:* The velocity profile for the Elrod-Adams model is computed under the additional assumption that the liquid phase of the lubricant is adhering to the moving surface (we recall that, actually, the Elrod-Adams model does not provide any information on the position of the liquid phase). Thus, the velocity at the free boundary (which separates the lubricant and the "empty" phase) is equal to 0. The velocity in the "empty" phase is reduced to 0. In details, we obtain:

$$\mu_l\,\frac{\partial^2 U}{\partial z^2}(x,z) = \frac{dp}{dx}(x), \ \text{ for } z \in]0, s(x)h(x)[$$

  with the boundary conditions $U(x,0) = v_0$ and $U(x, s(x)h(x)) = 0$. In the empty phase, we have $U(x,z) = 0$, for $z \in]s(x)h(x), h(x)[$.

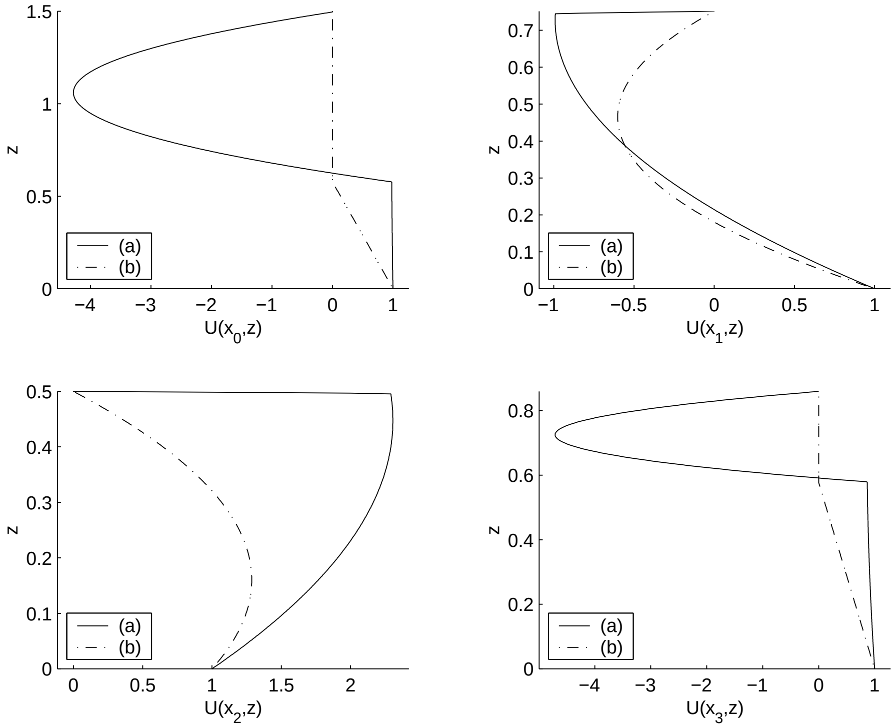


Figure 4.10. Horizontal velocity U at $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 4/5$, for the Buckley-Leverett model with $\varepsilon = 10^{-3}$ (a) and the Elrod-Adams model (b) (Case (i))

Case (ii): the liquid phase is adhering to the upper (fixed) surface

FIG.4.11 (resp. 4.12) gives the pressure (resp. saturation) distribution obtained with the Buckley-Leverett / Reynolds model for various values of ε. We can see that when ε decreases, the Buckley-Leverett saturation tends to the uniform distribution $s \equiv \theta_{in}$. The pressure tends to 0 uniformly. Therefore, the Buckley-Leverett / Reynolds model does not approach the Elrod-Adams model in that configuration: assuming that the lubricant is adhering to the upper (fixed) surface is not relevant. In fact, we can observe that, in this configuration, the shear effects tend to vanish, as it is pointed out further. FIG.4.13 describes the horizontal velocity distribution in the whole domain with the two models (Buckley-Leverett / Reynolds and Elrod-Adams).

- *Computation of the velocity for the Buckley-Leverett / Reynolds model:* Equations (4.31)–(4.33) still hold, up to this adapted definition (which takes into account the position of each phase):

$$\eta(x,z) = \begin{cases} \mu_g, & \text{if} \quad 0 < z < (1-s(x))h(x), \\ \mu_l, & \text{if} \quad (1-s(x))h(x) < z < h(x). \end{cases}$$

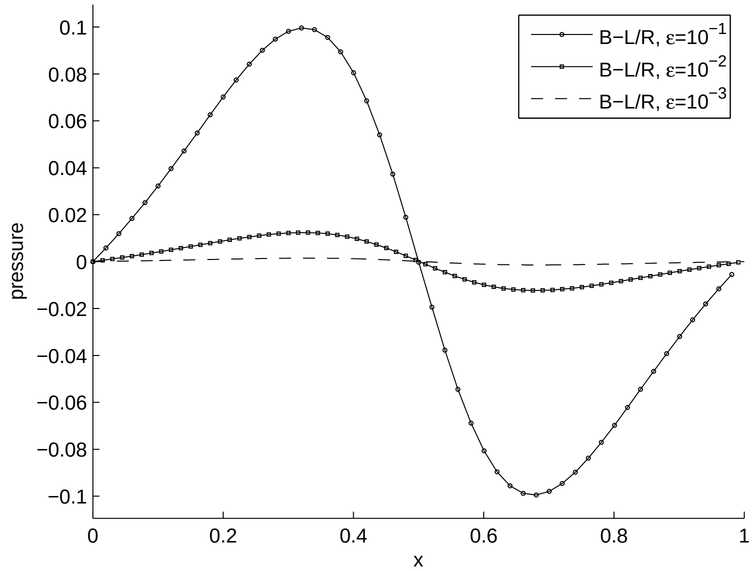


Figure 4.11. Pressure distribution for the Buckley-Leverett model (Case (ii))

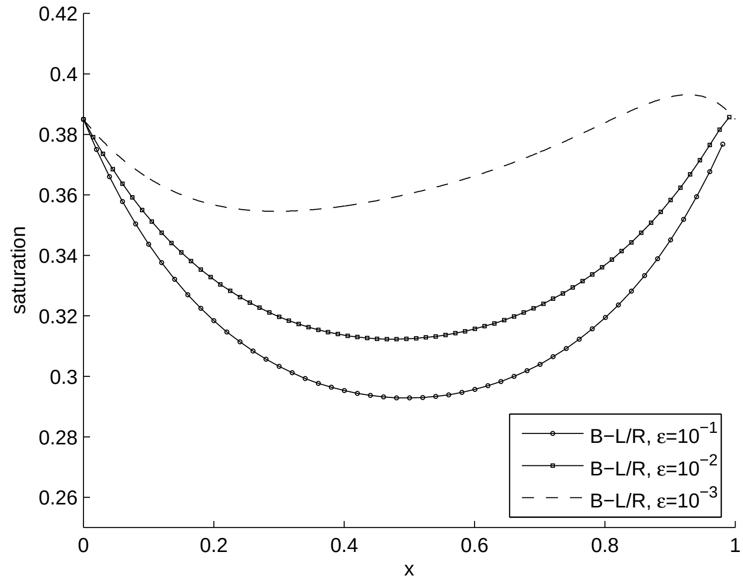


Figure 4.12. Saturation distribution for the Buckley-Leverett model (Case (ii))

Now, as in Case (i), this is completed with the following equality at the interface:

$$\lim_{z \to ((1-s(x))h(x))^-} \left\{ \mu_g \, \frac{\partial U}{\partial z}(x,z) \right\} = \lim_{z \to ((1-s(x))h(x))^+} \left\{ \mu_l \, \frac{\partial U}{\partial z}(x,z) \right\}.$$

- *Computation of the velocity for the Elrod-Adams model:* The velocity profile for the Elrod-Adams model is computed under the additional assumption that the liquid phase of the lubricant is adhering to the upper surface (which is not supposed by the model). We obtain:

$$\mu_l \, \frac{\partial^2 U}{\partial z^2}(x,z) = \frac{dp}{dx}(x), \text{ for } z \in](1-s(x))\, h(x), h(x)[,$$

with the boundary conditions $U(x,h(x)) = v_0$ and $U(x,(1-s(x))\, h(x)) = 0$. In the empty phase, we have $U(x,z) = 0$, for $z \in]0,(1-s(x))\, h(x)[$.

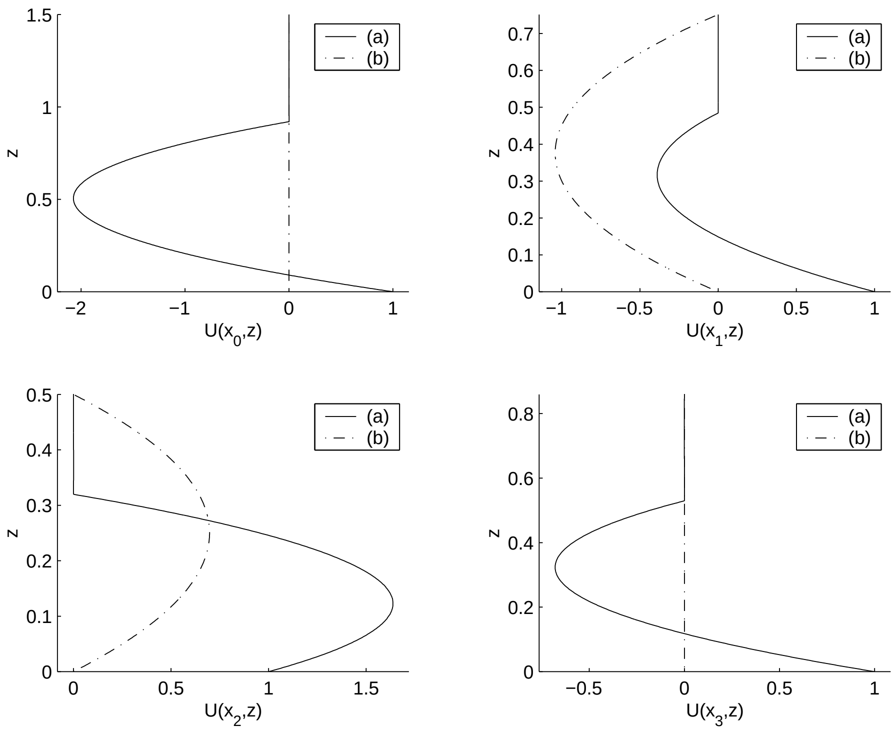


Figure 4.13. Horizontal velocity U at $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 4/5$, for the Buckley-Leverett model with $\varepsilon = 10^{-3}$ (a) and the Elrod-Adams model (b) (Case (ii))

Let us recall that the lubricant in liquid phase is considered as the reference fluid. Thus, velocity profiles show that the lubricant does not support the shear effects at all. This can be viewed also through the expression of the Buckley-Leverett flux, $Q_{in} f(s) +$

$v_0h(x)g(s)$, in which it is observed that when ε tends to 0, the non-classical contribution to the Buckley-Leverett flux, i.e. the shear contribution $v_0h(x)g(s)$, tends to vanish.

Appendix. Expression of the flux functions

For the sake of simplicity, let us introduce the following function:

$$\alpha_\varepsilon^{(i)}(s) = 1 - (1-\varepsilon)s^i, \quad i = 1, 2, 3.$$

Case (i): the lubricant is adhering to the lower (moving) surface

This corresponds to FIG.4.1. The functions A, B are given by:

$$\begin{aligned} A(s) &:= A_\varepsilon^{(1)}(s) = \frac{4\alpha_\varepsilon^{(1)}(s)\alpha_\varepsilon^{(3)}(s) - 3(\alpha_\varepsilon^{(2)}(s))^2}{\varepsilon\ \alpha_\varepsilon^{(1)}(s)}, \\ B(s) &:= B_\varepsilon^{(1)}(s) = \frac{\alpha_\varepsilon^{(2)}(s)}{\alpha_\varepsilon^{(1)}(s)}, \end{aligned}$$

and the flux functions f and g are defined by:

$$\begin{aligned} f(s) := f_\varepsilon^{(1)}(s) &= \varepsilon s^2 \frac{3(\alpha_\varepsilon^{(2)}(s))^2 - 2s\alpha_\varepsilon^{(1)}(s)}{4\alpha_\varepsilon^{(1)}(s)\alpha_\varepsilon^{(3)}(s) - 3(\alpha_\varepsilon^{(2)}(s))^2}, \\ g(s) := g_\varepsilon^{(1)}(s) &= -f_\varepsilon^{(1)}(s)\,\frac{\alpha_\varepsilon^{(2)}(s)}{2\alpha_\varepsilon^{(1)}(s)} + s\left(1 - \varepsilon\frac{s}{2\alpha_\varepsilon^{(1)}(s)}\right). \end{aligned}$$

Case (ii): the lubricant is adhering to the upper (fixed) surface

This corresponds to FIG.4.2. We emphasize that the flux functions or Reynolds coefficients can be written, in a simple way, as a perturbation of the ones described in the first case. Indeed, the functions A, B are given by:

$$\begin{aligned} A(s) &:= A_\varepsilon^{(2)}(s) = A_\varepsilon^{(1)}(s), \\ B(s) &:= B_\varepsilon^{(2)}(s) = B_\varepsilon^{(1)}(s) - \frac{2s(1-s)(1-\varepsilon)}{\alpha_\varepsilon^{(1)}(s)}, \end{aligned}$$

and the flux functions f and g are defined by:

$$\begin{aligned} f(s) := f_\varepsilon^{(2)}(s) &= f_\varepsilon^{(1)}(s), \\ g(s) := g_\varepsilon^{(2)}(s) &= g_\varepsilon^{(1)}(s) - \frac{s(1-s)}{\alpha_\varepsilon^{(1)}(s)}\left(1 - (1-\varepsilon)f_\varepsilon^{(2)}(s)\right). \end{aligned}$$

From the previous formulas, it can be deduced that only the shear terms are modified by the assumption on the geometrical assumption: indeed, in the Buckley-Leverett (resp. Reynolds) equation, only the flux function g (resp. right-hand side B) is modified.

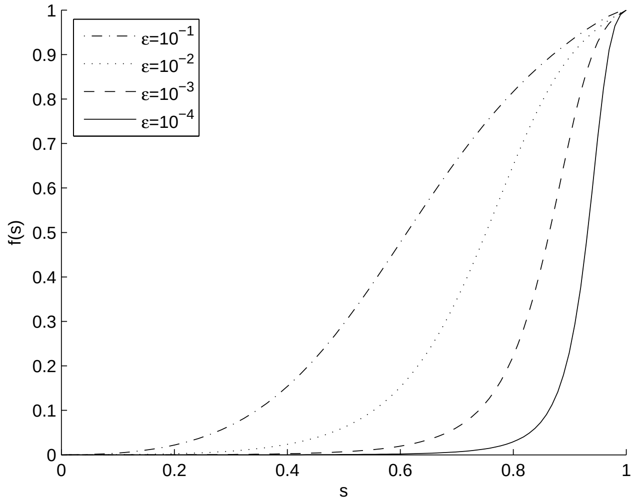


Figure 4.14. Classical contribution to the Buckley-Leverett flux function f (Cases $(i) - (ii)$)

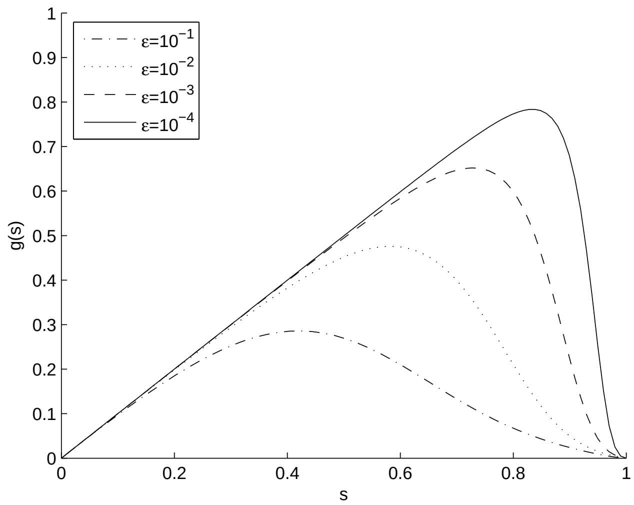


Figure 4.15. Shear contribution to the Buckley-Leverett flux function g (Case (i))

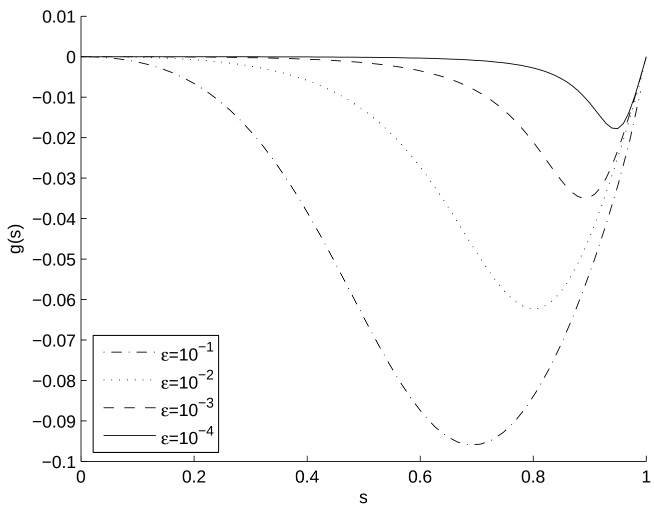


Figure 4.16. Shear contribution to the Buckley-Leverett flux function g (Case (ii))

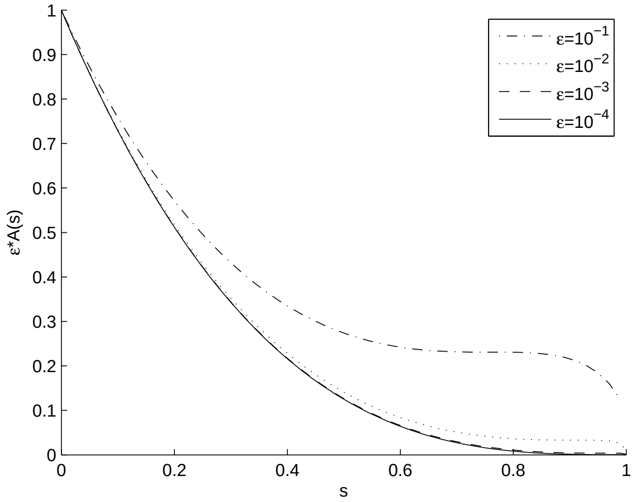


Figure 4.17. Left-hand side weight function in the Reynolds equation A (Cases $(i)-(ii)$)

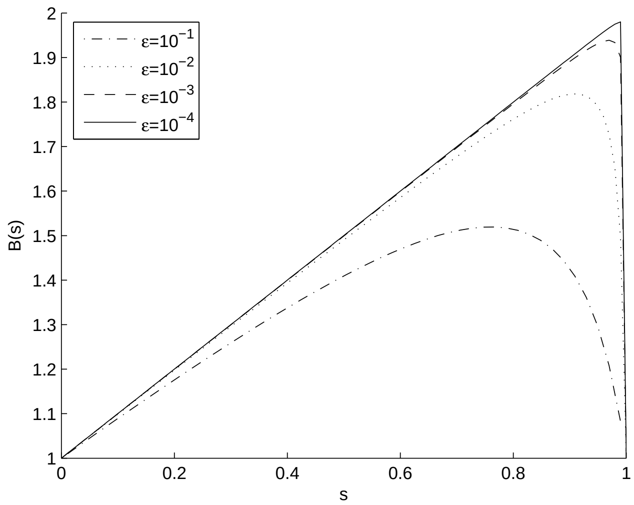


Figure 4.18. Right-hand side weight function in the Reynolds equation B (Case (i))

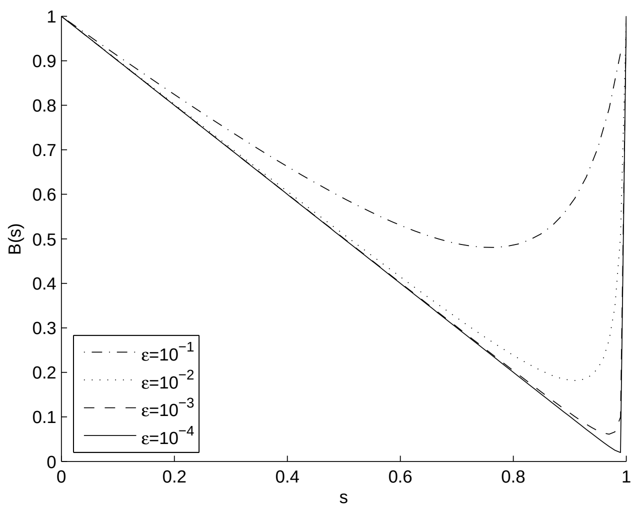


Figure 4.19. Right-hand side weight function in the Reynolds equation B (Case (ii))

Bibliography

[BC86] G. Bayada and M. Chambat. The transition between the Stokes equations and the Reynolds equation: a mathematical proof. *Appl. Math. Optim.*, 14(1):73–93, 1986.

[BC83] G. Bayada and M. Chambat. Analysis of a free boundary problem in partial lubrication. *Quart. Appl. Math.*, 40(4):369–375, 1982/83.

[BCG01] G. Bayada, M. Chambat, and S. R. Gamouana. About thin film micropolar asymptotic equations. *Quart. Appl. Math.*, 59(3):413–439, 2001.

[BLRN79] C. Bardos, A. Y. Le Roux, and J.-C. Nédélec. First order quasilinear equations with boundary conditions. *Comm. Partial Differential Equations*, 4(9):1017–1034, 1979.

[Car99] J. Carrillo. Entropy solutions for nonlinear degenerate problems. *Arch. Ration. Mech. Anal.*, 147(4):269–361, 1999.

[CC83] G. Capriz and G. Cimatti. Partial lubrication of full cylindrical bearings. *ASME J. Lubrication Technol.*, 105:84–89, 1983.

[CE70] J. Coyne and H. G. Elrod. Conditions for the rupture of a lubricating film, Part 1. *ASME J. Lubrication Technol.*, 92:451–456, 1970.

[CE71] J. Coyne and H. G. Elrod. Conditions for the rupture of a lubricating film, Part 2. *ASME J. Lubrication Technol.*, 93:156–167, 1971.

[DL88] F. Dubois and P. LeFloch. Boundary conditions for nonlinear hyperbolic systems of conservation laws. *J. Differential Equations*, 71(1):93–122, 1988.

[EA75] H. G. Elrod and M. L. Adams. A computer program for cavitation. *Cavitation and related phenomena in lubrication - Proceedings - Mech. Eng. Publ. ltd*, pages 37–42, 1975.

[Kru70] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[MNRR96] J. Málek, J. Nečas, M. Rokyta, and M. Ružička. *Weak and measure-valued solutions to evolutionary PDEs*, volume 13 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1996.

[NPD97] A. Nouri, F. Poupaud, and Y. Demay. An existence theorem for the multi-fluid Stokes problem. *Quart. Appl. Math.*, 55(3):421–435, 1997.

[Ott96] F. Otto. Initial-boundary value problem for a scalar conservation law. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(8):729–734, 1996.

[Pao03] L. Paoli. Asymptotic behavior of a two fluid flow in a thin domain: from Stokes equations to Buckley-Leverett equation and Reynolds law. *Asymptot. Anal.*, 34(2):93–120, 2003.

[Sab00] J. Sabil. Etude théorique et numérique d'un écoulement diphasique non-miscible et incompressible dans un domaine mince. *Rapport de Recherche du laboratoire MAPLY CNRS-UMR 5585 (Stage de D.E.A.)*, 2000.

[Ser96] D. Serre. *Systèmes de lois de conservation. II.* Fondations. [Foundations]. Diderot Editeur, Paris, 1996. Structures géométriques, oscillation et problèmes mixtes. [Geometric structures, oscillation and mixed problems].

[SJPT91] J. Saint Jean Paulin and K. Taous. About the derivation of Reynolds law from Navier-Stokes equation for two non-miscible fluids. in *Mathematical Modellings in Lubrication*, pages 99–104. Publicacions da Universidade de Vigo, Editors: G. Bayada, M. Chambat, J. Durany, Vigo, 1991.

[Tic96] J. A. Tichy. Non-newtonian lubrification with the convected Maxwell model. *ASME J. Tribol.*, 118:344–348, 1996.

[Vov02] J. Vovelle. Convergence of finite volume monotone schemes for scalar conservation laws on bounded domains. *Numer. Math.*, 90(3):563–596, 2002.

5

Conclusion

Nous proposons quelques développements qui s'inscrivent dans la continuité du travail présenté dans ce mémoire.

Homogénéisation pour des rugosités quelconques

Comme cela a été souligné dans les Chapitres 1 et 2, le problème homogénéisé pour des rugosités quelconques possède une structure différente de celle du problème initial. Si nous parvenons à démontrer l'existence d'une classe particulière de solutions avec saturation isotrope, il convient de souligner que ces résultats ne sont pas totalement satisfaisants car ils laissent en suspens le problème dans le cadre général et ne répondent pas à la question suivante : quelle est la structure du problème homogénéisé dans le cas général ? De manière sous-jacente se pose la question de l'unicité pour le problème homogénéisé que nous avons établi. Mais cette question est délicate et, par ailleurs, nous ne savons pas même démontrer l'unicité (éventuelle) d'une solution au sein de la classe des solutions isotropes (ce qui serait un critère satisfaisant). Ainsi, de nombreux travaux restent en cours afin d'améliorer les résultats obtenus. Le cas de rugosités obliques est, de ce point de vue, intéressant car il constitue un cas intermédiaire entre le cas général et le cas où le problème homogénéisé est bien posé. Il met à jour les effets d'anisotropie de l'écoulement, même s'il révèle l'existence de fonctions de saturation anisotropes difficiles à interpréter. L'une des idées, afin de généraliser ces résultats, consiste à tenter de déterminer des transformations $(x_1, x_2) \to (X_1, X_2)$ (non nécessairement isométriques) permettant de se ramener à des cas favorables au processus d'homogénéisation, après changement de variables.

L'homogénéisation du problème de lubrification a été effectuée afin de prendre en compte des rugosités périodiques, ce qui est valable pour de nombreux types de défauts (qui peuvent être obtenus, volontairement ou non, par les procédés de fabrication). Néanmoins, on peut s'interroger sur la prise en compte de rugosités non déterministes, à l'instar des travaux de Harp et Salant mentionnés dans l'Appendice B. Une telle étude nécessite la mise en oeuvre d'outils d'analyse stochastique et permettrait d'étudier l'influence des rugosités dont l'origine est davantage “ accidentelle ”.

Comportement asymptotique en temps d'une loi de conservation scalaire sur un domaine borné

Une caractéristique immédiate des profils de pression saturation dans le modèle d'Elrod-Adams est la discontinuité, à la rupture du film, de la saturation. Ce profil peut partiellement être obtenu à partir d'un processus évolutif (voir Chapitre 4) qui permet de montrer que la discontinuité de la saturation est un choc.
De manière générale, si l'on s'intéresse aux solutions stationnaires de l'équation de Buckley-Leverett généralisée, on peut montrer qu'il en existe une infinité (pour un certain choix de données). L'unique saturation stationnaire issue du processus entropique peut être une combinaison de deux solutions stationnaires, reliées par un choc, située en une position z. Ainsi, un des enjeux consiste à localiser la position stationnaire du choc (si elle existe). Pour cela, il est nécessaire d'étudier la structure des solutions stationnaires de lois de conservation scalaires sur un domaine borné. Le problème modèle considéré est alors le suivant :

$$\left\{ \begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(Qf(u) + H(x)g(u)\right) = 0, &\quad \text{sur } (0,+\infty)\times(0,1) \\ u(0,\cdot) = u_0, &\quad \text{sur } (0,1) \\ u = u^D, &\quad \text{sur } (0,T)\times\{0,1\} \end{aligned} \right.$$

avec les données suivantes :

- $u_0 \in L^\infty(0,1)$, $0 \le u_0 \le 1$ p.p. ; $u^D \in [0,1]$,
- $f \in C^1([0,1])$, f croissante, $f(0) = 0$, $f(1) = 1$, et $g(u) = u(1 - f(u))$,
- $H \in C^1([0,1])$, $H(x) = H(1-x)$, $H > 0$, H décroissante sur $[0,1/2]$,
- $Q > 0$.

Ce problème est une version simplifiée du problème de lubrification établi au Chapitre 4, mais il contient toutes les difficultés qui nous intéressent. En particulier, le choix de la donnée au bord u^D en corrélation avec celui du débit Q est important car il permet de distinguer plusieurs régimes aux propriétés très différentes et de localiser la présence (éventuelle) et la position d'un choc. Les travaux sont en cours de développement afin de décrire rigoureusement la structure de l'unique solution stationnaire issue du processus entropique. Ce sujet fait l'objet d'un travail en collaboration avec Julien Vovelle et fournit déjà quelques pistes intéressantes.

Comportement limite de l'équation de Buckley-Leverett : prise en compte d'un mélange liquide / vide

Les simulations présentées au Chapitre 4 avec le modèle bifluide ont été effectuées avec un rapport de viscosité réaliste $\varepsilon = 10^{-3}$, modélisant un mélange liquide-gaz. Le modèle d'Elrod-Adams, classiquement utilisé en lubrification, ne s'intéresse qu'à la phase liquide : soit la zone est complètement saturée, soit elle est partiellement saturée auquel cas la deuxième phase peut être assimilée... à du vide ! Dans cette perspective, il faudrait, pour mieux comparer les deux modèles, prendre un rapport de viscosité égal à $\varepsilon = 0$ dans le modèle bifluide. Or, nous avons vu que pour de petits rapports de viscosité, des difficultés importantes surgissent, d'un point de vue numérique et théorique. Théoriquement, les coefficients du modèle bifluide tendent à dégénérer. Plutôt que de s'intéresser à ces problèmes délicats, nous proposons deux approches :

1/ La première approche est de déterminer les équations limites pour les équations de Buckley-Leverett / Reynolds généralisées, lorsque ε tend vers 0. Pour cela, on peut considérer à nouveau le problème simplifié précédent, avec un flux partiel f de la forme :
$$f(s) = \max\left(1 + \frac{s-1}{\varepsilon}, 0\right)$$

2/ la deuxième serait d'utiliser les résultats issus de l'analyse du comportement asymptotique en temps de l'équation de Buckley-Leverett, obtenu dans le cadre décrit précédemment (deuxième point de cette conclusion). Le but est de caractériser la position du choc lorsque ε tend vers 0 et de démontrer, par exemple, que le résultat obtenu coincide avec la position de la discontinuité (rupture du film) dans le modèle d'Elrod-Adams.

Ainsi, ce travail s'inscrit dans la continuité du sujet présenté ci-dessus, en collaboration avec Julien Vovelle.

A

Homogenization of the dam problem

ABSTRACT *This appendix deals with the homogenization of the dam problem, as formulated by Brezis, Kinderlehrer and Stampacchia [BKS78] and Alt [Alt79]. This study is motivated by the varying permeability which allows to take into account a layer structure, for instance. This has been partially studied by Rodrigues [Rod84] and, here, we give additional results.*

A.1 Introduction to the dam problem

The dam problem has first been stated using variational inequalities by Baiocchi and co-authors [Bai80, BCMP73, BF77] and Benci [Ben74]. But this approach is only possible for dams with vertical walls (typically rectangular dams). The formulation of the dam problem for domains with general shapes has been introduced by Brezis, Kinderlehrer and Alt Stampacchia [BKS78, Alt79]. Let us consider a bounded domain, denoted Ω, with a locally Lipschitz continuous boundary $\partial\Omega = \Gamma \cup \Gamma_0 \cup \Gamma_a$ (see FIG. A.1). Γ is an impervious boundary, Γ_0 is the boundary in contact with the open air and Γ_a is the boundary in contact with water. The basic problem is to find the pressure p and the fluid saturation θ in Ω. Introducing the permeability of the porous medium, denoted k, the strong formulation, based on Darcy's law ([Dar56] for historical references), is given by the following set of equations:

$$\begin{cases} -\nabla \cdot \Big(k(x)\ \nabla p(x)\Big) - \dfrac{\partial}{\partial x_2}\Big(k(x)\ \theta(x)\Big) = 0, & x \in \Omega \\ p(x) \geq 0, \quad p(x)(1-\theta(x)) = 0, \quad 0 \leq \theta(x) \leq 1, & x \in \Omega \end{cases}$$

with the boundary conditions:

$$\begin{cases} p(x) = p_a(x), & x \in \Gamma_a \\ p(x) = 0, & x \in \Gamma_0 \\ -k(x)\ \nabla p(x) \cdot \nu(x) - \theta(x)\ k(x)\ \nu_2(x) \geq 0, & x \in \Gamma_0 \\ -k(x)\ \nabla p(x) \cdot \nu(x) - \theta(x)\ k(x)\ \nu_2(x) = 0, & x \in \Gamma \end{cases}$$

with, for example, $p_a(x) = \begin{cases} 0, & \text{on } \Gamma_0 \\ h_i - x_2, & \text{on } \Gamma_{a,i} \end{cases}$, where h_i denotes the water level in the i-th reservoir.

Remark A.1. *In the earlier strong formulation, one should notice the boundary condition on Γ_0 corresponding to a sign condition on the fluid flow: it is designed to eliminate the non physical solutions and it means that no water flows into the dam through Γ_0.*

Remark A.2. *Similarly to the dam problem, it is possible to define a generalized lubrication problem, i.e. with a speed direction $e_\gamma = (\cos\gamma, \sin\gamma)$ instead of $e = (1,0)$ in the classical problem:*

$$\begin{cases} -\nabla \cdot \Big(h^3(x)\ \nabla p(x)\Big) = -\nabla \cdot \Big(\theta(x) h(x)\ e_\gamma\Big), & x \in \Omega \\ p(x) \geq 0, \quad p(x)\ (1 - \theta(x)) = 0, \quad 0 \leq \theta(x) \leq 1, & x \in \Omega \end{cases}$$

with the boundary conditions:

$$\begin{cases} p(x) = 0, & x \in \Gamma_0 \\ p(x) = p_a(x_1), & x \in \Gamma_a \\ \Big(\theta h\ e_\gamma - h^3\ \nabla p\Big) \cdot \nu \geq 0, & x \in \Gamma_0 \\ p \text{ and } \Big(\theta h\ e_\gamma - h^3\ \nabla p\Big) \cdot \nu \text{ are } 2\pi x_1 \text{ periodic} \end{cases}$$

In the previous formulation, a sign condition has been added (in comparison with the classical problem with $e = (1,0)$): indeed the flow condition means that no water should flow into the domain through Γ_0. Interestingly when considering the classical direction $e = (1,0)$, one gets:

$$h^3 \frac{\partial p}{\partial x_2} \geq 0, \qquad \text{on} \quad \Gamma_0$$

which is trivially satisfied since $p_{|\Gamma_0} = 0$ and $p \geq 0$ a.e. on Ω. Thus we did not need to take into account this flow condition in the initial problem (with $e = (1,0)$). We can even show that this condition is trivially satisfied when $\sin\gamma \leq 0$. But in the most general case, it has to be imposed.

From now on let us consider that the following assumptions are satisfied:

Assumption 20.

$$\exists\ k_0, k_1, \quad \forall\ x \in \Omega, \quad 0 < k_0 \leq k(x) \leq k_1$$

Assumption 21.

(i) p_a *is a Lipschitz continuous function on* $\Gamma_0 \cup \Gamma_a$*,*

(ii) $p_a = 0$ *on* Γ_0 *and* $p_a > 0$ *on* Γ_a*.*

The formulation of the problem, existence and uniqueness of a solution (under the earlier assumptions) have been studied in particular by Brezis, Kinderlehrer and Stampacchia [BKS78], Alt [Alt79], Chipot and Carrillo [CC82, Chi84].

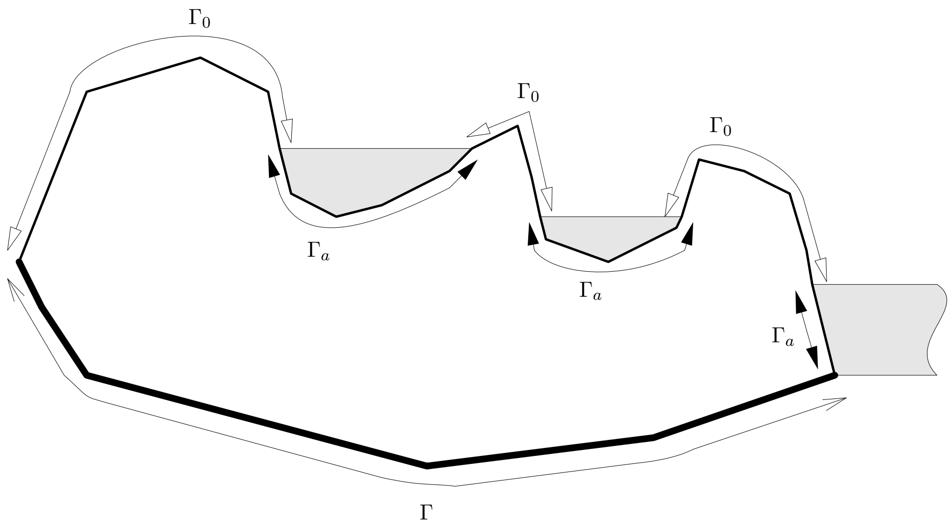


Figure A.1. Dam domain

The variational formulation of that problem is given by:

$$(\mathcal{P}_d)\begin{cases} \text{Find } (p,\theta) \in V_a \times L^\infty(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega k\ \nabla p\ \nabla\phi + \int_\Omega \theta\ k\ \frac{\partial\phi}{\partial x_2} \leq 0, \quad \forall\ \phi \in V_0 \\ p \geq 0, \quad p\ (1-\theta) = 0, \quad 0 \leq \theta \leq 1, \quad \text{a.e.} \end{cases}$$

with the following functional spaces:

$$\begin{aligned} V_a &= \Big\{ v \in H^1(\Omega),\ v_{|\Gamma_0} = 0,\ v_{|\Gamma_a} = p_a \Big\} \\ V_0 &= \Big\{ v \in H^1(\Omega),\ v_{|\Gamma_0} \geq 0,\ v_{|\Gamma_a} = 0 \Big\} \end{aligned}$$

Similarly to the lubrication problem, it is possible to define a penalized dam problem with an approximate Heaviside function $H_\eta(p(x))$ playing the role of the saturation function θ.

A.2 Homogenization results

The homogenization process given in the previous sections is useful to describe the behaviour of a porous medium. When the permeability of the medium is highly oscillating, it can be described as:

$$k_\varepsilon(x) = k\left(x, \frac{x}{\varepsilon}\right), \quad \forall x \in \Omega$$

We suppose that k_ε satisfies Assumption 20 uniformly with respect to ε. With the same methods used in Chapter 1, it is easy to establish the behaviour of the fluid flow when ε goes to 0. For convenience, proofs are omitted.

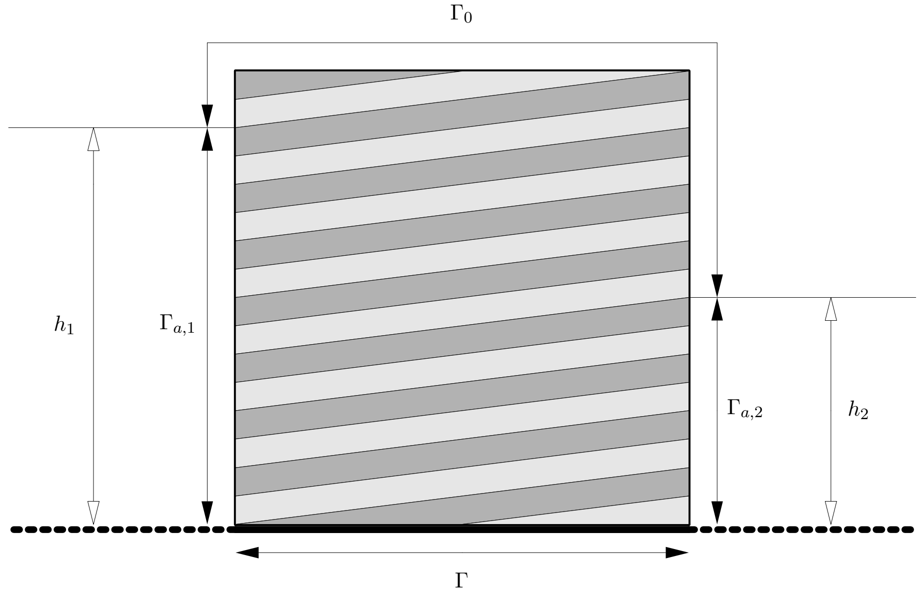


Figure A.2. Rectangular dam with layers

From now on let us consider a dam with layers (see Fig. A.2). Let us suppose that :

Assumption 22. *The permeability k_ε of the porous medium can be written as:*

$$k_\varepsilon(x) = k_1\left(x, \frac{X_1^\gamma(x)}{\varepsilon}\right) \; k_2\left(x, \frac{X_2^\gamma(x)}{\varepsilon}\right) \tag{A.1}$$

where the mapping $x \rightarrow X^\gamma(x)$ denotes the rotation of angle γ, that is:

$$\left\{\begin{array}{lcr} X_1^\gamma(x) & = & \cos\gamma\ x_1 + \sin\gamma\ x_2 \\ X_2^\gamma(x) & = & -\sin\gamma\ x_1 + \cos\gamma\ x_2 \end{array}\right.$$

We should notice that $\gamma = 0$ (or $\gamma = \pi/2$) corresponds to horizontal and/or vertical layers.

Theorem A.3. *Under Assumption 22, the homogenized dam problem is defined by:*

$$(\mathcal{P}_d^\star)\left\{\begin{array}{l} \text{Find } (p_0,\ \Theta_1,\ \Theta_2) \in V_a \times L^\infty(\Omega) \times L^\infty(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \mathcal{A}\cdot\nabla p_0\ \nabla\phi \leq \int_\Omega b_1^0\ \frac{\partial\phi}{\partial x_1} + \int_\Omega b_2^0\ \frac{\partial\phi}{\partial x_2}, \quad \forall\ \phi \in V_0 \\ p_0 \geq 0, \quad p_0\ (1-\Theta_i) = 0, \quad 0 \leq \Theta_i \leq 1, \quad (i=1,2), \quad \text{a.e.} \end{array}\right.$$

with the following expressions:

$$\mathcal{A}(x) = \begin{pmatrix} \kappa_1^\star(x) & 0 \\ & \\ 0 & \kappa_2^\star(x) \end{pmatrix} + \ (\kappa_1^\star(x) - \kappa_2^\star(x))\ \sin\gamma \begin{pmatrix} -\sin\gamma & \cos\gamma \\ & \\ \cos\gamma & \sin\gamma \end{pmatrix}$$

$$b_1^0(x) = \Big(\kappa_1^\star(x)\ \Theta_1(x) - \kappa_2^\star(x)\ \Theta_2(x)\Big)\ \sin\gamma\ \cos\gamma$$

$$b_2^0(x) = \Big(\kappa_1^\star(x)\ \Theta_1(x) - \kappa_2^\star(x)\ \Theta_2(x)\Big)\ \sin^2\gamma\ +\ \kappa_2^\star(x)\ \Theta_2(x)$$

The homogenized coefficients are defined by $\kappa_i^\star = \dfrac{\widetilde{k_j}}{\widetilde{\widetilde{k_i^{-1}}}}$ *($i,j = 1,2$ and $i \neq j$).*

Moreover problem $(\mathcal{P}_d^\star)$ admits $(p_0,\ \Theta_1,\ \Theta_2)$ as a solution where (p_0, θ_0) is the two-scale limit of $(p_\varepsilon, \theta_\varepsilon)$ (solution of problem $(\mathcal{P}_d^\varepsilon)$) and with the following notations :

$$\Theta_i(x) = \Big[\frac{1}{\widetilde{k_j}}\widetilde{\Big(\theta_0 k_j\Big)}\Big](x), \quad i,j = 1,2, \quad j \neq i$$

Remark A.4. *Similarly to the lubrication problem, Theorem A.3 gives an example of an homogenized problem with non-diagonal terms in the matrix of the first member and additional homogenized coefficients in the second member. Indeed, let us try to understand the homogenized problem in a form that is close to the initial problem. Then we will define* *main terms and residual terms as follows:*

$$\mathcal{A} = \underbrace{\begin{pmatrix} \kappa_1^\star(x) & 0 \\ 0 & \kappa_2^\star(x) \end{pmatrix}}_{A^m} \underbrace{+ \left(\kappa_1^\star(x) - \kappa_2^\star(x)\right)\ \sin\gamma \begin{pmatrix} -\sin\gamma & \cos\gamma \\ \cos\gamma & \sin\gamma \end{pmatrix}}_{A^r}$$

where A^m (resp. A^r) can be identified as the main (resp. residual) matrix.

$$b_1^0(x) = \underbrace{\Big(\kappa_1^\star(x)\ \Theta_1(x) - \kappa_2^\star(x)\ \Theta_2(x)\Big)\ \sin\gamma\ \cos\gamma}_{b_1^r}$$

$$b_2^0(x) = \underbrace{\Big(\kappa_1^\star(x)\ \Theta_1(x) - \kappa_2^\star(x)\ \Theta_2(x)\Big)\ \sin^2\gamma}_{b_2^r} + \underbrace{\kappa_2^\star(x)\ \Theta_2(x)}_{b_2^m}$$

Let us notice that the main term in the second member only appears in the x_2 direction, since the gravity force is vertical. When neglecting the residual terms defined earlier (A^r, b_1^r, b_2^r) in the formulation of Theorem A.3, the homogenized problem can be put under a form that is similar to the initial one (with anisotropic coefficients). Theorem A.3 with $\gamma = 0$ gives us the homogenized dam problem with vertical or horizontal layers.

Corollary A.5. *The homogenized dam problem, for $\gamma = 0$, is defined by:*

$$(\mathcal{P}_d^\star)\left\{\begin{array}{l} \text{Find } (p_0,\ \Theta) \in V_a \times L^\infty(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \mathcal{A}\cdot\nabla p_0\ \nabla\phi \leq \int_\Omega \Theta\ \kappa_2\ \frac{\partial\phi}{\partial x_2}, \quad \forall\ \phi \in V_0 \\ p_0 \geq 0, \quad p_0\ (1-\Theta) = 0, \quad 0 \leq \Theta \leq 1, \quad \text{a.e.} \end{array}\right.$$

with $\mathcal{A} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$. The homogenized coefficients are defined by $\kappa_i^\star = \dfrac{\widetilde{k_j}}{\widetilde{k_i^{-1}}}$.
Moreover problem $(\mathcal{P}_d^\star)$ admits $(p_0,\ \Theta)$ as a solution where (p_0, θ_0) is the two-scale limit of $(p_\varepsilon, \theta_\varepsilon)$ (solution of problem $(\mathcal{P}_d^\varepsilon)$) and with the following notations :

$$\Theta(x) = \Big[\frac{1}{\widetilde{k_1}}\widetilde{\Big(\theta_0 k_1\Big)}\Big](x)$$

Remark A.6. *Uniqueness of the solution $(p_0,\ \Theta)$ can be obtained under additional assumptions: with the terminology used by Chipot [Chi84], "if the shape of Ω is such that no pool appears" and if the homogenized coefficients are constant (see [CR81]), then the solution is unique (it is quite easy to make a change of coordinates so that we get an homogeneous dam problem) and the homogenized saturation Θ is a characteristic function.*

Bibliography

[Alt79] H. W. Alt. Strömungen durch inhomogene poröse Medien mit freiem Rand. *J. Reine Angew. Math.*, 305:89–115, 1979.

[Bai80] C. Baiocchi. Free boundary problems in fluid flow through porous media and variational inequalities. in *Free boundary problems, Vol. I (Pavia, 1979)*, pages 175–191. Ist. Naz. Alta Mat. Francesco Severi, Rome, 1980.

[BCMP73] C. Baiocchi, V. Comincioli, E. Magenes, and G.A. Pozzi. Free boundary problems in the theory of fluid flow through porous media: existence and uniqueness theorems. *Ann. Mat. Pura Appl. (4)*, 97:1–82, 1973.

[Ben74] V. Benci. On a filtration problem through a porous medium. *Ann. Mat. Pura Appl. (4)*, 100:191–209, 1974.

[BF77] C. Baiocchi and A. Friedman. A filtration problem in a porous medium with variable permeability. *Ann. Mat. Pura Appl. (4)*, 114:377–393, 1977.

[BKS78] H. Brezis, D. Kinderlehrer, and G. Stampacchia. Sur une nouvelle formulation du problème de l'écoulement à travers une digue. *C. R. Acad. Sci. Paris Sér. A-B*, 287(9):A711–A714, 1978.

[CC82] J. Carrillo and M. Chipot. On the dam problem. *J. Differential Equations*, 45(2):234–271, 1982.

[Chi84] M. Chipot. *Variational inequalities and flow in porous media*, volume 52 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1984.

[CR81] M. Codegone and J.-F. Rodrigues. On the homogenization of the rectangular dam problem. *Rend. Sem. Mat. Univ. Politec. Torino*, 39(2):125–136 (1982), 1981.

[Dar56] H. Darcy. *Les fontaines publiques de la ville de Dijon.* Dalmont, Paris, first edition, 1856.

[Rod84] J.-F. Rodrigues. Some remarks on the homogenization of the dam problem. *Manuscripta Math.*, 46(1-3):65–82, 1984.

B

An average flow model of the Reynolds roughness with a mass-flow preserving cavitation model

Article paru dans
ASME Journal of Tribology

ABSTRACT *An average Reynolds equation for predicting the effects of deterministic periodic roughness, taking JFO mass flow preserving cavitation model into account, is introduced based upon double scale analysis approach. This average Reynolds equation can be used both for a microscopic interasperity cavitation and a macroscopic one. The validity of such a model is verified by numerical experiments both for one dimensional and two dimensional roughness patterns.*

NOMENCLATURE

A_ε, B_ε, A_i, B_i	=	partial differential operators
$a^\star_{ij}$, $a^\star_i$, a^0_i	=	auxiliary homogenized coefficients
$A^\star_{ij}$, $B^\star_i$, B^0_i	=	homogenized coefficients
h_1, h_2	=	description of the gap
h, h_ε	=	actual gap
h_s	=	smooth part of the gap
h_r	=	amplitude of the roughness
p	=	pressure
p_0, p_1...	=	approximations of the pressure
Q	=	input flow value

U	=	velocity
$x = (x_1, x_2)$	=	dimensionless space coordinates
$y = (y_1, y_2)$	=	microscale coordinates
$X = (X_1, X_2)$	=	oblique coordinates
$X' = (X'_1, X'_2)$	=	real coordinates
$Y =]0,1[\times]0,1[$	=	rescaled microcell
γ	=	obliqueness angle
$\partial/\partial n$	=	normal derivative
μ	=	viscosity
ε	=	roughness spacing
θ	=	saturation
θ_0	=	microscopic homogenized saturation
Θ, Θ_1,Θ_2	=	macrohomogenized saturations
w_i, χ_i^0	=	auxiliary functions defined on Y
$\overline{\cdot}^Y$	=	average operator with respect to y
$[\cdot]_{Y_1}$	=	average operator with respect to y_1
$[\cdot]_{Y_2}$	=	average operator with respect to y_2

B.0 Introduction

The effects of the surface roughness on the behavior of a thin film flow has long been the subject of intensive studies. Various ways have been introduced to study Reynolds roughness by seeking an average equation with smooth coefficients. Some of the most popular results are the Christensen formula [CT71] for longitudinal and transverse roughness and the Patir and Cheng flow factor model [PC78] for a more general surface roughness pattern. Two wide classes of results can be outlined. In the first one, which is deterministic, a periodic description of the surfaces is often assumed to be known and linked to a specific process of the surface [SSar]. It is possible to distinguish macrovariables and microvariables and to use a mathematical homogenization approach to rigorously obtain an average Reynolds equation by making the period of the roughness tend to zero [BF89]. The coefficients of this average Reynolds equation implicitly contain the description of the microroughness elementary cell. The second class of results deals with a statistical description of the surface roughness. Following the Patir and Cheng approach, numerous authors proposed an average Reynolds equation in which the coefficients included the knowledge of the surface statistics by way of flow factors which can be evaluated by numerical experiments. Rigorously speaking, this approach is less satisfactory than the first one, assuming *a priori* the existence of a control volume in which the average flow rates can be equivalently expressed in terms of flow factors. The number and quantities (Peklenik number, combined root mean square roughness...) involved in the characterization of the flow factors can also be discussed. Moreover, as the initial Reynolds equation, the average Reynolds equation can be expressed in terms of $\nabla \cdot (K\nabla p) = F$ in which K is a diagonal matrix. This seems to be contradictory with the result obtained by the first approach in which K is a non diagonal matrix for two dimensional general roughness pattern [JBS02].

Up to now, these averaging processes never take cavitation into account. A common procedure is to use the average equation instead of the classical Reynolds equation with Gumbel and Swift Steiber boundary conditions or to include it in the S.O.R. algorithm proposed by Richardson, thus obtaining the splitting of the lubricated device in two areas. In a first area, the pressure is greater than the cavitation pressure and the average Reynolds equation is valid; in the other area, pressure is equal to the cavitation pressure. It is well known [DMT84, BC86, BC88], however, that none of these models is mass preserving, especially through the cavitation area. Jakobsson, Floberg and Olsson (JFO) [FJ57, Ols65] developped a set of conditions for the cavitation boundary that properly takes the conservation of mass into account in the whole device. Elrod [EA75, Elr81] proposed a slightly modified formulation and a related specific algorithm. The mathematical related problem evidences a hyperbolic-parabolic feature which renders difficult both theoretical study and numerical experiments [BC86, PS92, Bre86, VK90]. It is the goal of this paper to develop in a rigorous way an average JFO Reynolds equation for the deterministic periodic roughness pattern. So far, few papers have been devoted to such a problem. Recently, the interasperity cavitation has been studied by way of a statistical approach [KCNO80, HS01]. The Patir and Cheng flow factor method is extended and an average Reynolds equation is proposed. The resulting equation has the same left-hand side that in the Patir and Cheng equation (cavitation has no effect on the corresponding flow factors) while the right-hand side of the equation is modified and new flow factors are introduced. At last Harp and Salant [HS01] proposed to modify the boundary conditions by a value which is a function of the wavelength of the roughness. Our approach is quite different and explicitly based upon the introduction of fast and slow variables. The initial equation is rewritten in terms of these two variables and asymptotic expansion of the pressure is introduced with respect to a small parameter associated to the roughness wavelength. The goal is to find an equation satisfied by the first terms of the expansion. Some assumptions about the shape of the roughness appear to be necessary to solve the problem, leading to a new average Reynolds cavitation equation. This equation has numerous common features with the initial Reynolds equation: it is also a two unknowns pressure-saturation formulation. Some particular cases - transverse, diagonal and longitudinal roughness - will be studied in details.

B.1 Basic equations

Our studied cavitation model, like the Elrod algorithm and its variants, views the film as a mixture. It does not, however, make the assumption of liquid compressibility in the full film area as in [VK90] and some other papers. As in [KB91, SS00], only the liquid-vapor mixture in the cavitated region is assumed compressible. The flow obeys the following "universal" Reynolds equation (here written in a dimensionless form) through all the gap in which the pressure cavitation is assumed to be zero in the cavitation area

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(h^3 \frac{\partial p}{\partial x_i}\right) = \frac{\partial \theta h}{\partial x_1}, \tag{B.1}$$

$$p \geq 0, \tag{B.2}$$

$$0 \leq \theta \leq 1, \tag{B.3}$$

$$p\,(1-\theta) = 0. \tag{B.4}$$

In this steady state isoviscous version of the equation, p is the pressure, θ is the relative mixture density, h the film thickness, x_1 is the direction of the effective relative velocity of the shaft, while x_2 is the transverse direction.

This system of equations can be understood as follows (see [BC86, FJ57, Ols65, EA75, Elr81, Bre86, KB91] for various comments and meaning of the θ variable):

- the well-known Reynolds equation holds in the full film region, that is $p > 0$ and $\theta = 1$,
- a mass flow conserving equation $\partial\theta h/\partial x_1 = 0$ holds in the cavitated region with $p = 0$ and $0 < \theta < 1$.
- a boundary condition which is also mass flow preserving at the (unknown) interface between the two regions:

$$-h^3 \frac{\partial p}{\partial n} + h\cos(n, x_1) = \theta h \cos(n, x_1).$$

The reason to retain this specific cavitation equation is that it has been the subject of numerous mathematical studies [BC86] giving a strong and rigorous basis to the following manipulations [BMV05]. To be noticed, however, that our approach can be applied without difficulty to other cavitation models as the one in [VK90]. Last, it has to be mentioned that this equation takes both macrocavitation (associated to the occurrence of a diverging part of a bearing for example) and interasperity cavitation into account.

The boundary conditions depend on the considered device. However, the following ones are often used, corresponding for example to a journal bearing with an axial supply groove. The pressure is imposed at two circumferential locations and one axial location. The last boundary condition is an input flow condition at the axial location corresponding to the supply groove:

$$\theta(x)h(x) - h^3(x)\frac{\partial p}{\partial x_1}(x) = Q. \tag{B.5}$$

For small values of Q, starvation may occur in the vicinity of the supply groove.

B.2 Asymptotic expansion

Let us suppose that the roughness is periodically reproduced in the two x_1 and x_2 directions from an elementary cell Y (or "miniature bearing" in Tonder's terminology). We denote by ε the ratio of the homothetic transformation passing from the elementary cell $Y = Y_1 \times Y_2$ to the real bearing and by $y_1 = x_1/\varepsilon$ and $y_2 = x_2/\varepsilon$ the local variables (see FIG. B.1).

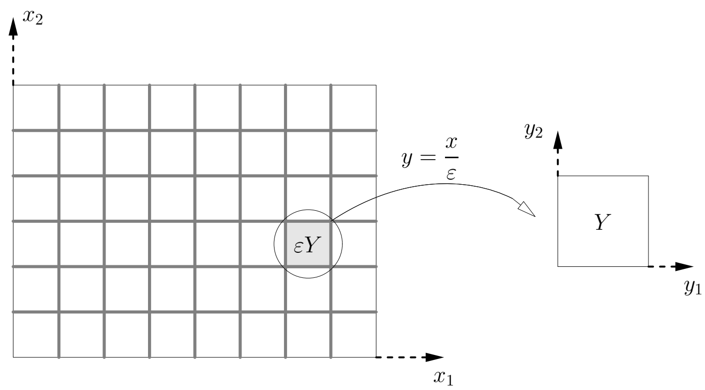


Figure B.1. Macroscopic domain and elementary cells

Let us now consider shapes that can be written $h_\varepsilon(x) = h(x, x/\varepsilon)$. We suppose furthermore that they are described by

$$h_\varepsilon(x) = h_1\left(x, \frac{x_1}{\varepsilon}\right) h_2\left(x, \frac{x_2}{\varepsilon}\right)$$

which allows us to take into account either transverse or longitudinal roughnesses, but also more general two dimensional roughnesses. Introducing now the fast variables y_1 and

y_2, it appears that the new expression for the gap is:

$$h(x,y) = h_1(x, y_1)\, h_2(x, y_2)\,. \tag{B.6}$$

The combined computation in terms of (x_1, x_2) or (y_1, y_2) is an important feature of the method. It is convenient to consider first x and y as independent variables and to replace next y by x/ε (see [BF89]).

B.2.1 Formulation of average equations

We denote by A_ε the initial differential Reynolds operator

$$A_\varepsilon[\cdot] = \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(h^3\left(x, \frac{x}{\varepsilon}\right)\frac{\partial\,[\cdot]}{\partial x_j}\right),$$

and we also define the right-hand side operator

$$B_\varepsilon[\cdot] = \frac{\partial}{\partial x_1}\left(h\left(x, \frac{x}{\varepsilon}\right)[\cdot]\right).$$

The Reynolds equation (B.1) becomes

$$A_\varepsilon(p) = B_\varepsilon(\theta).$$

The underscript ε indicates the dependance of the real pressure on the microtexture related to ε. We also define the following operators:

$$\begin{aligned}
A_1[\cdot] &= \sum_{j=1}^{2} \frac{\partial}{\partial y_j}\left(h^3(x,y)\frac{\partial\,[\cdot]}{\partial y_j}\right),\\
A_2[\cdot] &= \sum_{j=1}^{2} \frac{\partial}{\partial y_j}\left(h^3(x,y)\frac{\partial\,[\cdot]}{\partial x_j}\right) + \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(h^3(x,y)\frac{\partial\,[\cdot]}{\partial y_j}\right),\\
A_3[\cdot] &= \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(h^3(x,y)\frac{\partial\,[\cdot]}{\partial x_j}\right),
\end{aligned}$$

and also

$$\begin{aligned}
B_1^i[\cdot] &= \frac{\partial}{\partial y_i}\left(h(x,y)\ [\cdot]\ \right), \quad i = 1, 2,\\
B_2^i[\cdot] &= \frac{\partial}{\partial x_i}\left(h(x,y)\ [\cdot]\ \right), \quad i = 1, 2.
\end{aligned}$$

If applied to a function of $(x, x/\varepsilon)$, the operators become

$$A_\varepsilon = \left(1/\varepsilon^2\ A_1 + 1/\varepsilon\ A_2 + A_3\right), \tag{B.7}$$

$$B_\varepsilon = \left(1/\varepsilon\ B_1^1 + B_2^1\right). \tag{B.8}$$

We shall look for an asymptotic expansion of the solutions

$$p(x) = p_0(x, \frac{x}{\varepsilon}) + \varepsilon p_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 p_2\left(x, \frac{x}{\varepsilon}\right) + ..., \tag{B.9}$$

$$\theta(x) = \theta_0\left(x, \frac{x}{\varepsilon}\right), \tag{B.10}$$

each unknown p_i and θ_0 being a function of (x, y). The problem of the boundary conditions to be satisfied by the p_i is somewhat difficult but may be summarized as follows.

(i) The natural boundary conditions on $(p_\varepsilon, \theta_\varepsilon)$ are assigned to p_0 and an equivalent saturation linked to θ_0, which will be developped in next subsection.

(ii) The function p_i, $i \geq 1$, are Y periodic, i.e. periodic in the two variables y_1, y_2, for each value of (x_1, x_2).

To be noticed that unlike of p, we do not introduce an asymptotic expansion for θ. This can be explained by observing the evolution of p and θ as ε tends to 0 (see FIG.B.2 for instance). Clearly, the oscillations of the pressure are decreasing and p tends to a smooth function (namely p_0 which, actually, does not depend on the fast variable as it will be pointed out further). This is not the case for θ and an asymptotic smooth limit cannot be considered.

We shall see later that the functions p_i, $i \geq 1$, are defined up to an additive constant. Moreover, from Equations (B.2)–(B.4), the following properties hold:

$$p_0(x, y) \geq 0, \tag{B.11}$$

$$0 \leq \theta_0(x, y) \leq 1, \tag{B.12}$$

$$p_0(x, y)\ (1 - \theta_0(x, y)) = 0. \tag{B.13}$$

Putting Equations (B.9) and (B.10) into Equation (B.1) and taking account of Equations (B.7) and (B.8), one can write by an identification procedure:

$$A_1 p_0 = 0, \tag{B.14}$$

$$A_1 p_1 + A_2 p_0 = B_1^1 \theta_0, \tag{B.15}$$

$$A_1 p_2 + A_2 p_1 + A_3 p_0 = B_2^1 \theta_0. \tag{B.16}$$

Let us remark that these equations are of the following type: *For a given F, find a function q, depending on the variable y, q being Y periodic, such that (x is a parameter),*

$$A_1 q = F. \tag{B.17}$$

A condition to have a solution for Equation (B.17) is

$$\int_Y F(x,y)\ dy = 0. \tag{B.18}$$

Moreover, if q is a solution, then $q + c$ with c any constant with respect to y is also a solution. Applying Condition (B.18) to Equation (B.14), we deduce that p_0 does not depend on y

$$p_0(x). \tag{B.19}$$

Let us suppose now that p_0 is known, and noticing that, due to boundary conditions, $(B_1^1\theta_0 - A_2 p_0)$ satisfies Equation (B.18), existence of p_1 is guaranteed. Now we can represent p_1 as a function of p_0 in a more suitable form. We define w_i and χ_i^0 $(i = 1, 2)$ as the Y periodic solutions (up to an additive constant) of the following local problems:

$$A_1\ w_i = \frac{\partial h^3}{\partial y_i}, \quad i = 1, 2, \tag{B.20}$$

$$A_1\ \chi_i^0 = \frac{\partial \theta_0 h}{\partial y_i}, \quad i = 1, 2. \tag{B.21}$$

The solution of Equation (B.15) reduces to

$$p_1(x,y) = \chi_1^0(x,y) - \frac{\partial p_0}{\partial x_1}(x) w_1(x,y) - \frac{\partial p_0}{\partial x_2}(x) w_2(x,y). \tag{B.22}$$

The same procedure can be used to ensure the existence of p_2, but in that step, the corresponding condition (B.18) applied to Equation (B.16) becomes

$$\int_Y (B_2^1\theta_0 - A_2 p_1 - A_3 p_0)\ dy = 0. \tag{B.23}$$

Then the main idea is to put Equation (B.22) into Equation (B.23), so that the only remaining unknowns are p_0 and θ_0.

By analogy with the probabilistic framework, we denote by $\overline{u}^Y$ the local average of any Y periodic function u:

$$\overline{u}^Y(x) = \frac{1}{[Y]} \int_Y u(x,y)\ dy.$$

By exchanging the integral and the derivation symbols, and after some calculations, Equation (B.23) becomes

$$\sum_{i,j} \frac{\partial}{\partial x_i}\left(A^\star_{ij}\frac{\partial p_0}{\partial x_j}\right) = \left(\frac{\partial B^0_1}{\partial x_1} + \frac{\partial B^0_2}{\partial x_2}\right), \tag{B.24}$$

where ($i, j = 1, 2$ and $j \neq i$)

$$\begin{aligned} A^\star_{ii} &= \overline{h^3}^Y - \overline{h^3\frac{\partial w_i}{\partial y_i}}^Y, \\ A^\star_{ij} &= -\overline{h^3\frac{\partial w_j}{\partial y_i}}^Y = -\overline{h^3\frac{\partial w_i}{\partial y_j}}^Y = A^\star_{ji}, \end{aligned}$$

and also

$$\begin{aligned} B^0_1 &= \overline{\theta_0 h}^Y - \overline{h^3\frac{\partial \chi^0_1}{\partial y_1}}^Y, \\ B^0_2 &= -\overline{h^3\frac{\partial \chi^0_1}{\partial y_2}}^Y. \end{aligned}$$

Equation (B.24) deals with any periodic roughness pattern. To be noticed is the fact that the differential operator is no more of the Reynolds type since extra terms $\partial^2 p_0/\partial x_i\partial x_j$ appear. The right-hand side also contains an additive term in the x_2 direction. However, the link between p_0 and θ_0 is not so clear. This is a major obstacle which prevents from getting a tractable equation. Nevertheless, Assumption B.6 allows us to solve the following difficulties:

- Computation of $A^\star_{ii}$, $i = 1, 2$:
  Let us recall Equation (B.20) with $i = 1$:

$$\frac{\partial}{\partial y_1}\left(h^3\frac{\partial w_1}{\partial y_1}\right) + \frac{\partial}{\partial y_2}\left(h^3\frac{\partial w_1}{\partial y_2}\right) = \frac{\partial h^3}{\partial y_1}.$$

  Since $h^3\partial w_1/\partial y_2$ is Y periodic, averaging this equation over Y_2 gives

$$\frac{\partial}{\partial y_1}\left(\left[h^3\frac{\partial w_1}{\partial y_1}\right]_{Y_2}\right) = \frac{\partial\left[h^3\right]_{Y_2}}{\partial y_1},$$

  where $[\cdot]_{[Y_i]}$ is the averaging operator over Y_i (for $i = 1, 2$).
  Thus we have, by integrating in the y_1 variable and using Equations(B.6):

$$\left[h^3 - h^3\frac{\partial w_1}{\partial y_1}\right]_{Y_2} = C_1,$$

where C_1 is a constant with respect to y. Let us notice that, averaging the earlier equation over Y_1 simply gives $C_1 = A^\star_{11}$. Thus, it remains to calculate C_1. Dividing each side of the previous equation by h_1^3:

$$\left[h_2^3\right]_{Y_2} - \left[h_2^3 \frac{\partial w_1}{\partial y_1}\right]_{Y_2} = \frac{C_1}{h_1^3}$$

and, since w_1 is Y periodic, averaging over Y_1 gives

$$A^\star_{11} = \frac{\overline{h_2^3}^Y}{\overline{h_1^{-3}}^Y}. \tag{B.25}$$

Following the same procedure, we state:

$$A^\star_{22} = \frac{\overline{h_1^3}^Y}{\overline{h_2^{-3}}^Y}. \tag{B.26}$$

- Computation of $A^\star_{ij}$, $i \neq j$:
Starting from Equation (B.20) with $i = 1$, since $h^3 - h^3\partial w_1/\partial y_1$ is Y periodic, averaging this equation over Y_1 gives

$$\frac{\partial}{\partial y_2}\left(\left[h^3 \frac{\partial w_1}{\partial y_2}\right]_{Y_1}\right) = 0.$$

Thus we have, by integrating in the y_2 variable

$$\left[h^3 \frac{\partial w_1}{\partial y_2}\right]_{Y_1} = C_2,$$

where C_2 is a constant with respect of y. Similarly to the computation of $A^\star_{ii}$, one has $C_2 = A^\star_{12} = A^\star_{21}$. Dividing each side of the equation by h_2^3:

$$\frac{C_2}{h_2^3} = \left[h_1^3 \frac{\partial w_1}{\partial y_2}\right]_{Y_1},$$

and, since w_1 is Y periodic, averaging over Y_2 gives $C_2\overline{a_1^{-1}}^Y = 0$, i.e.

$$A^\star_{12} = A^\star_{21} = 0. \tag{B.27}$$

Now, it remains to calculate the right-hand side of the Reynolds equation.

- Computation of B_1^0:
Let us recall Equation (B.21) with $i = 1$:

$$\frac{\partial}{\partial y_1}\left(h^3\frac{\partial \chi_1^0}{\partial y_1}\right) + \frac{\partial}{\partial y_2}\left(h^3\frac{\partial \chi_1^0}{\partial y_2}\right) = \frac{\partial \theta_0 h}{\partial y_1}.$$

Since $h^3\partial\chi_1^0/\partial y_2$ is Y periodic, averaging this equation over Y_2 gives

$$\frac{\partial}{\partial y_1}\left(\left[h^3\frac{\partial \chi_1^0}{\partial y_1}\right]_{Y_2}\right) = \frac{\partial\left[\theta_0 h\right]_{Y_2}}{\partial y_1}.$$

Thus we have, by integrating in the y_1 variable:

$$\left[\theta_0 h - h^3\frac{\partial \chi_0^1}{\partial y_1}\right]_{Y_2} = C_3,$$

where C_3 is a constant with respect to y. Clearly, we have $C_3 = B_1^0$. Dividing each side of the equation by h_1^3:

$$\left[\frac{\theta_0 h}{h_1^3}\right]_{Y_2} - \left[h_2^3\frac{\partial \chi_1^0}{\partial y_1}\right]_{Y_2} = \frac{C_3}{h_1^3},$$

and, since χ_1^0 is Y periodic, averaging over Y_1 gives $\overline{\theta_0 h/h_1^3}^{Y} = C_3\overline{h_1^{-3}}^{Y}$, i.e.

(B.28)
$$B_1^0 = \frac{\overline{\left(\dfrac{\theta_0 h_2}{h_1^2}\right)}^{Y}}{\overline{h_1^{-3}}^{Y}}$$

- Computation of B_2^0:
Starting from Equation (B.21) with $i = 1$, since the function $h^3 - h^3\partial\chi_1^0/\partial y_1$ is Y periodic, averaging this equation over Y_1 gives

$$\frac{\partial}{\partial y_2}\left(\left[h^3\frac{\partial \chi_1^0}{\partial y_2}\right]_{Y_1}\right) = 0.$$

Thus we have, by integrating in the y_2 variable:

$$-\left[h^3\frac{\partial \chi_1^0}{\partial y_2}\right]_{Y_1} = C_4,$$

where C_4 is a constant with respect of y. We have $C = B_2^0$. Then dividing each side by h_2^3:

$$\frac{C_4}{h_2^3} = \left[h_1^3 \frac{\partial \chi_1^0}{\partial y_2} \right]_{Y_1},$$

and, since χ_1^0 is Y periodic, averaging over Y_2 gives $C_4 \overline{h_1^{-3}}^Y = 0$, i.e.

$$B_2^0 = 0. \tag{B.29}$$

Now, it is obvious that Equation (B.24) can be written in a more simple way by using Equations (B.25)–(B.29). Before that, let us write the term B_1^0 in a more usable form. Defining the quantities

$$B_1^\star = \frac{\overline{h_1^{-2}}^Y}{\overline{h_1^{-3}}^Y} \overline{h_2}^Y, \tag{B.30}$$

$$\Theta = \frac{1}{\overline{h_2}^Y \overline{h_1^{-2}}^Y} \overline{\left(\frac{\theta_0 h_2}{h_1^2} \right)}^Y, \tag{B.31}$$

we get $B_1^0 = \Theta B_1^\star$. Moreover, from Equations (B.12) and (B.13), we immediately have:

$$0 \le \Theta(x) \le 1, \tag{B.32}$$

$$p_0(x)\ (1 - \Theta(x)) = 0, \tag{B.33}$$

so that the homogenized equations appear to be

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(A_{ii}^\star \frac{\partial p_0}{\partial x_i} \right) = \frac{\partial \Theta B_1^\star}{\partial x_1}, \tag{B.34}$$

$$p_0 \ge 0, \tag{B.35}$$

$$0 \le \Theta \le 1, \tag{B.36}$$

$$p_0\ (1 - \Theta) = 0, \tag{B.37}$$

where $A_{11}^\star$, $A_{22}^\star$ and $B_1^\star$ are, respectively, given by Equations (B.25), (B.26) and (B.30). Moreover, the link between a new (smooth) "macroscopic" saturation Θ and the (oscillating) "microscopic" saturation θ_0 is given by Equation (B.31). As an important feature, Θ is not the average of the microscopic saturation θ_0.

B.2.2 Average boundary condition

When the pressure is imposed, the corresponding average boundary condition is assigned to p_0. When an input flow is given on a supply line, the average flow condition is obtained following the asymptotic expansion method. Taking account of roughness patterns, Equation (B.5) becomes:

$$\theta(x)h\left(x, \frac{x}{\varepsilon}\right) - h^3\left(x, \frac{x}{\varepsilon}\right)\frac{\partial p}{\partial x_1}(x) = Q. \tag{B.38}$$

Putting Equations (B.9) and (B.10) into Equation (B.38), one can write by an identification procedure:

$$\theta_0(x,y)h(x,y) - h^3(x,y)\left(\frac{\partial p_0}{\partial x_1}(x) + \frac{\partial p_1}{\partial y_1}(x,y)\right) = Q.$$

Putting Equation (B.22) into it gives

$$\left(\theta_0 h - h^3\frac{\partial \chi_1^0}{\partial y_1}\right) - \left(h^3 - h^3\frac{\partial w_1}{\partial y_1}\right)\frac{\partial p_0}{\partial x_1} + \left(h^3\frac{\partial w_2}{\partial y_1}\right)\frac{\partial p_0}{\partial x_2} = Q.$$

Averaging over Y gives the boundary condition relating p_0 and Θ at the supply groove:

$$B_1^0 - A_{11}^\star\frac{\partial p_0}{\partial x_1} - A_{12}^\star\frac{\partial p_0}{\partial x_2} = Q,$$

and since $A_{12}^\star = 0$ and $B_1^0 = \Theta B_1^\star$, one gets:

$$\Theta B_1^\star - A_{11}^\star\frac{\partial p_0}{\partial x_1} = Q. \tag{B.39}$$

The next subsection deals with two main particular cases: transverse or longitudinal roughness.

B.2.3 Particular cases

- Transverse roughness: when the roughness does not depend on y_2, we have the homogenized equation, easily deduced from Equations (B.25)–(B.31)

$$\frac{\partial}{\partial x_1}\left(\frac{1}{\overline{h^{-3}}^Y}\frac{\partial p_0}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\overline{h^3}^Y\frac{\partial p_0}{\partial x_2}\right) = \frac{\partial}{\partial x_1}\left(\Theta\frac{\overline{h^{-2}}^Y}{\overline{h^{-3}}^Y}\right),$$

  with $\Theta = \dfrac{1}{\overline{h^{-2}}^Y}\overline{\left(\dfrac{\theta_0}{h^2}\right)}^Y$ and the boundary condition at the supply groove, deduced from Equation (B.39), should be read as:

$$\Theta\frac{\overline{h^{-2}}^Y}{\overline{h^{-3}}^Y} - \frac{1}{\overline{h^{-3}}^Y}\frac{\partial p_0}{\partial x_1} = Q.$$

■ Longitudinal roughness: when the roughness does not depend on y_1, we get

$$\frac{\partial}{\partial x_1}\left(\overline{h^3}^Y \frac{\partial p_0}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{\overline{h^{-3}}^Y}\frac{\partial p_0}{\partial x_2}\right) = \frac{\partial}{\partial x_1}\left(\Theta \overline{h}^Y\right),$$

with $\Theta = \dfrac{\overline{\theta_0 h}^Y}{\overline{h}^Y}$, and the boundary condition at the supply groove should be read as:

$$\Theta \overline{h}^Y - \overline{h^3}^Y \frac{\partial p_0}{\partial x_1} = Q.$$

B.3 Oblique roughness

Let us consider gaps that can be written as:

$$h_\varepsilon(x) = h_1\left(x, \frac{X_1(x)}{\varepsilon}\right) h_2\left(x, \frac{X_2(x)}{\varepsilon}\right),$$

with

$$\begin{cases} X_1(x) &= \quad \cos\gamma\ x_1 + \sin\gamma\ x_2, \\ X_2(x) &= -\sin\gamma\ x_1 + \cos\gamma\ x_2, \end{cases}$$

which allows us to take into account oblique roughness (with $h_2 \equiv 1$ for instance). The idea is to introduce a change of coordinates so that the assumption of Section B.2 on the roughness form in the new coordinates system is valid. The first step is to rewrite Equation (B.1) in the X coordinates:

$$\sum_{i=1}^{2} \frac{\partial}{\partial X_i}\left(h_\varepsilon^3 \frac{\partial p}{\partial X_i}\right) = \left(\frac{\partial \theta h_\varepsilon}{\partial X_1}\ \cos\gamma - \frac{\partial \theta h_\varepsilon}{\partial X_2}\ \sin\gamma\right).$$

Working now in the X coordinates and using the operators defined in Section B.2 (up to the writing in the X coordinates), we apply the asymptotic expansion technique to the earlier equation. With the formal asymptotic expansion used in Section B.2, we have in the (X, y) coordinates (with $y = X/\varepsilon$):

$$A_1 p_0 = 0, \tag{B.40}$$

$$A_1 p_1 + A_2 p_0 = B_1^1 \theta_0 \cos\gamma - B_1^2 \theta_0 \sin\gamma, \tag{B.41}$$

$$A_1 p_2 + A_2 p_1 + A_3 p_0 = B_2^1 \theta_0 \cos\gamma - B_2^2 \theta_0 \sin\gamma. \tag{B.42}$$

As in Section B.2, p_0 only depends on the X variable. Equation (B.41) allows us to determine p_1:

$$\begin{aligned} p_1(X,y) &= \chi_1^0(X,y)\cos\gamma - \chi_2^0(X,y)\sin\gamma \\ &\quad -w_1(X,y)\frac{\partial p_0}{\partial X_1}(X) - w_2(X,y)\frac{\partial p_0}{\partial X_2}(X). \end{aligned}$$

Then, putting this expression into Equation (B.42) gives:

$$\begin{aligned} \sum_{i,j}\frac{\partial}{\partial X_i}\left(a_{ij}^\star\frac{\partial p_0}{\partial X_j}\right) &= \frac{\partial}{\partial X_1}\left(b_{11}^0\cos\gamma + b_{12}^0\sin\gamma\right) \\ &+ \frac{\partial}{\partial X_2}\left(b_{21}^0\cos\gamma + b_{22}^0\sin\gamma\right), \end{aligned}$$

where the coefficients, which are easily computed as in Section B.2, are given by ($i, j = 1, 2$, $i \neq j$):

$$a_{ii}^\star = \overline{h^3}^Y - \overline{h^3\frac{\partial w_i}{\partial y_i}}^Y = \frac{\overline{h_j^3}^Y}{\overline{h_i^{-3}}^Y}, \tag{B.43}$$

$$a_{ij}^\star = -\overline{h^3\frac{\partial w_j}{\partial y_i}}^Y = 0, \tag{B.44}$$

and also ($i, j = 1, 2$, $i \neq j$)

$$b_{ii}^0 = \overline{\theta_0 h}^Y - \overline{h^3\frac{\partial \chi_i^0}{\partial y_i}}^Y = \frac{1}{\overline{h_i^{-3}}^Y}\overline{\left(\frac{\theta_0 h_j}{h_i^2}\right)}^Y, \tag{B.45}$$

$$b_{ij}^0 = -\overline{h^3\frac{\partial \chi_i^0}{\partial y_j}}^Y = 0. \tag{B.46}$$

As in Section B.2, defining the quantities

$$b_i^\star = \frac{\overline{h_i^{-2}}^Y}{\overline{h_i^{-3}}^Y}\overline{h_j}^Y, \tag{B.47}$$

$$\Theta_i = \frac{1}{\overline{h_j}^Y\,\overline{h_i^{-2}}^Y}\overline{\left(\frac{\theta_0 h_j}{h_i^2}\right)}^Y, \tag{B.48}$$

one has $b_{ii}^0 = \Theta_i\, b_i^\star$, with $p_0(1-\Theta_i) = 0$ and $0 \leq \Theta_i \leq 1$.

Finally, going back to the initial x coordinates, one gets the following homogenized problem:

$$\sum_{i,j} \frac{\partial}{\partial x_i}\left(A^\star_{ij}\frac{\partial p_0}{\partial x_j}\right) = \left(\frac{\partial B^0_1}{\partial x_1} + \frac{\partial B^0_2}{\partial x_2}\right), \tag{B.49}$$

$$p_0 \geq 0, \tag{B.50}$$

$$0 \leq \Theta_i \leq 1, \quad (i = 1, 2), \tag{B.51}$$

$$p_0\,(1 - \Theta_i) = 0, \quad (i = 1, 2), \tag{B.52}$$

with the left hand-side coefficients:

$$A^\star_{11} = a^\star_{11} - (a^\star_{11} - a^\star_{22}) \sin^2\gamma,$$

$$A^\star_{22} = a^\star_{22} + (a^\star_{11} - a^\star_{22}) \sin^2\gamma,$$

$$A^\star_{12} = A^\star_{21} = (a^\star_{11} - a^\star_{22}) \sin\gamma\cos\gamma,$$

and the right hand-side member:

$$\begin{aligned} B^0_1 &= \Theta_1 b^\star_1 - (\Theta_1 b^\star_1 - \Theta_2 b^\star_2)\sin^2\gamma, \\ B^0_2 &= (\Theta_1 b^\star_1 - \Theta_2 b^\star_2)\sin\gamma\ \cos\gamma, \end{aligned}$$

the coefficients $a^\star_{ii}$, $b^\star_i$ $(i = 1, 2)$ being given by Equation (B.43)[1] and Equation (B.47)[1] in the x coordinates. The link between the "microscopic saturation" θ_0 and the two "macroscopic saturations" Θ_i $(i = 1, 2)$ is given by Equation (B.48)[1].

At first glance, Equation (B.49) is very similar to (B.24). A major difference however is the anisotropic aspect of the saturation with two saturation functions Θ_i $(i = 1, 2)$, one for each direction.

From a mathematical point of view, it is not clear wether the system of Equations (B.49)–(B.52) is a closed one or not: is a supplementary equation needed to obtain a well-posed problem or not? Nevertheless, it can be proved that $\Theta_1 = \Theta_2$ is a possible choice for a solution of the system. With this assumption, it is possible to solve Equations (B.49)–(B.52) by using the same kind of algorithms as the ones used to solve Equations (B.1)–(B.4). The only difference lies in the modified coefficients and the fact that the direction of the flow is no longer the x_1 axis but an oblique one.

[1] to be translated in the x coordinates

B.4 Numerical results

As both Equations (B.1) and (B.24) have the same mathematical feature, various algorithms (see [DMT84, BC86, EA75, PS92, VK90, KB91, Hoo98]) used to compute solutions of Equations (B.1)–(B.4) can be adressed for the solution of Equation (B.24). In this paper, we propose the characteristics method adapted to steady state problems to deal with nonlinear convection term combined with finite elements. Moreover, the nonlinear Elrod-Adams model for cavitation is treated by a duality method. The combination of these numerical techniques has been explained and successfully applied by Bayada, Chambat and Vazquez in [BCV98].

B.4.1 Computation of homogenized coefficients

We consider effective gaps defined with either transverse or longitudinal roughness patterns. Let us recall that homogenized coefficients are given in Table 1.1 (see page 95).

The coefficients corresponding to Assumption B.6 can be easily obtained from the ones that are presented in Table 1.1, using products taking account of roughness effects in each direction. The coefficients corresponding to oblique roughness can be obtained by using products and linear combinations of coefficients given in Table 1.1.

B.4.2 Transverse roughness tests

We adress the numerical simulation of journal bearing devices with axial supply of lubricant. Indeed we simulate a journal bearing device whose length is denoted L, the mean radius $R_m = (R_b + R_j)/2$, R_b and R_j being the bearing and journal radii respectively, and the clearance is $c = R_b - R_j$. The supply flow is Q_R, the lubricant viscosity is μ and the velocity of the journal is U. Moreover, the roughless gap between the two surfaces is given by:

$$H_s(X') = c\left(1 + \rho\ \cos\left(\frac{X_1'}{R_m}\right)\right), \quad X' = (X_1', X_2') \in (0, 2\pi R_m) \times (0, L)$$

where the eccentricity ρ satisfies $0 \leq \rho < 1$. The classical Reynolds problem, in real variables $X' = (X_1', X_2')$, should be posed as follows:

$$\nabla \cdot \Big(\frac{H_s^3}{6\mu}\nabla P\Big) = U\frac{\partial}{\partial X_1'}\Big(\theta H_s\Big) \tag{B.53}$$

$$P \geq 0, \quad 0 \leq \theta \leq 1, \quad P\,(1-\theta) = 0, \tag{B.54}$$

with the boundary conditions

$$P = 0, \tag{B.55}$$

except on the supply groove in which

$$U\theta H_s - \frac{H_s^3}{6\mu}\frac{\partial P}{\partial X_1'} = Q_R. \tag{B.56}$$

Now let us introduce the dimensionless coordinates and quantities that provide the effective system to be solved:

$$\begin{array}{ccc} x_1 = \dfrac{X_1'}{R_m}, & x_2 = \dfrac{X_2'}{R_m}, & h_s = \dfrac{H_s}{c}, \\ p = \dfrac{c^2}{6\mu U R_m}\,P, & Q = \dfrac{Q_R}{cU}, & \kappa = \dfrac{R_m}{L}. \end{array}$$

Then, the dimensionless Reynolds problem becomes for $x \in (0, 2\pi) \times (0, \kappa)$:

$$\nabla \cdot \Big(h_s^3 \nabla p\Big) = \frac{\partial}{\partial x_1}\Big(\theta h_s\Big), \tag{B.57}$$

$$p \geq 0, \quad 0 \leq \theta \leq 1, \quad p\,(1-\theta) = 0, \tag{B.58}$$

with the boundary conditions

$$\theta h_s - h_s^3 \frac{\partial p}{\partial x_1} = Q, \tag{B.59}$$

on the boundary corresponding to the dimensionless supply groove (namely $\{0\} \times (0, \kappa)$), and the condition

$$p = 0, \tag{B.60}$$

on the other boundaries. The roughless gap is now $h_s(x) = 1 + \rho\ \cos(x_1)$. For the numerical tests, we have worked on the dimensionless equations, with the following data:

- $\kappa = 1$, i.e. $R_m = L$.
- The domain being $(0, 2\pi) \times (0, 1)$, the rough dimensionless gap is given by:

  $$h(x, x/\varepsilon) = h_s(x) + h_r(x/\varepsilon) = 1 + \rho\cos(x_1) + (1-\rho)\tau \sin\left(2\pi\frac{x_1}{\varepsilon}\right),$$

  with $\rho = 0.75$, $\tau = 0.7$, h_s (respectively h_r) denoting the smooth (respectively rough) contribution to the gap.

- The dimensionless flow at the supply groove is $Q = \theta_{in} h_s(0)$ with $\theta_{in} = 0.4571$.

For various values of ε, FIG.B.2 represents the behavior of both pressure and saturation. In particular, it justifies the formal asymptotic expansion used in Section B.2. Three main facts have to be observed:

1/ The oscillations of the pressure tend to vanish, thus showing that p tends to a smooth limit pressure (i.e. $p_0(x)$).

2/ The oscillations of the saturation do not vanish; the gradient tends to explode. Thus, $\theta(x)$ behaves like a function which depends on both slow and fast variables (i.e. $\theta_0(x, y)$).

3/ The existence of two cavitation areas at both extremities of the bearing (starvation phenomenon).

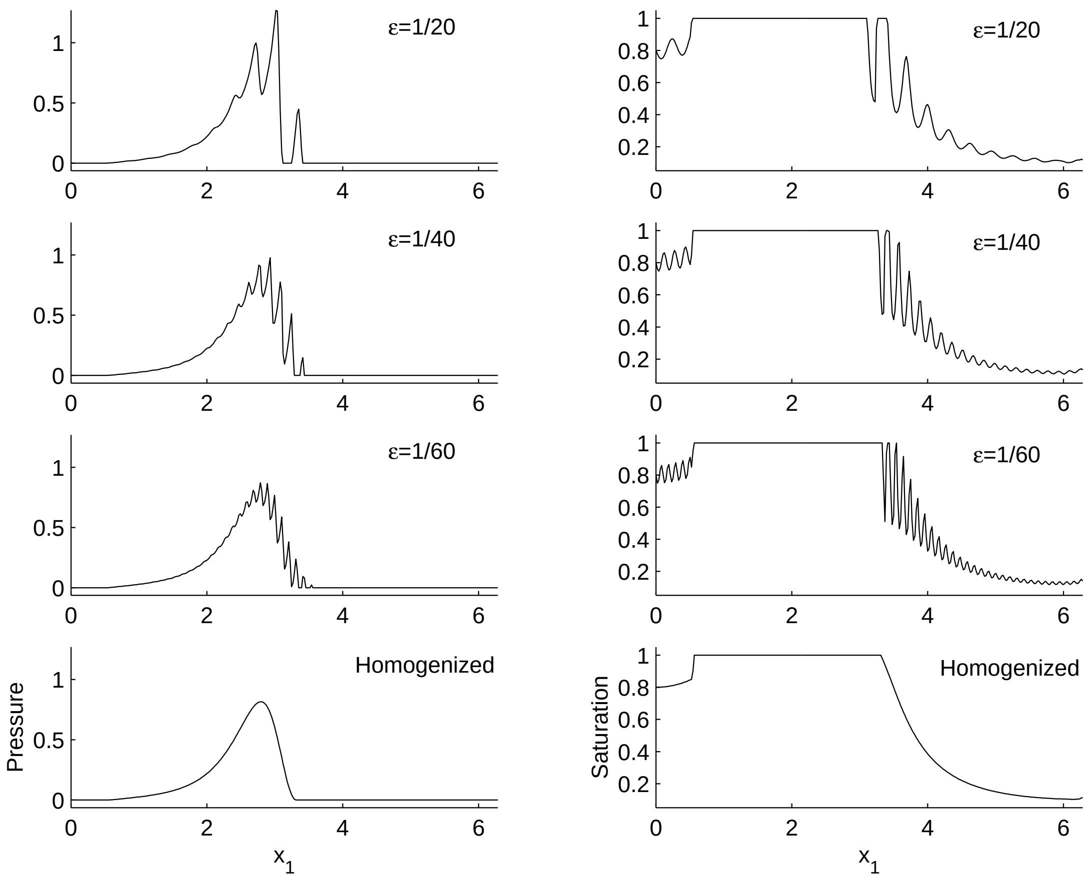


Figure B.2. Pressure and saturation at $x_2 = 0.5$ for different roughness periods

FIG.B.2 allows us not only to compare more precisely the convergence of the pressure to the homogenized one, but also to observe the behaviour of the saturation. The homogenized saturation may be viewed as an average, with respect to y, of the microsaturation weighted by roughness parameters.

B.4.3 Two dimensional roughness effects

The only difference with the previous subsection lies in the definition of the dimensionless gap $h(x, y)$ defined by Assumption B.6, other data being unchanged:

$$\begin{aligned} h_1(x, y_1) &= 1 + 0.5\cos(x_1) + 0.35\sin(2\pi y_1)\,, \\ h_2(x, y_2) &= 1 + 0.35\sin(2\pi y_2)\,. \end{aligned}$$

FIG.B.3 represents the pressure at a fixed x_1 (notice that the corresponding saturation figure is omitted, since there is nearly no cavitation).

FIG.B.4 and B.5 represent pressure and saturation at a fixed x_2, for various values

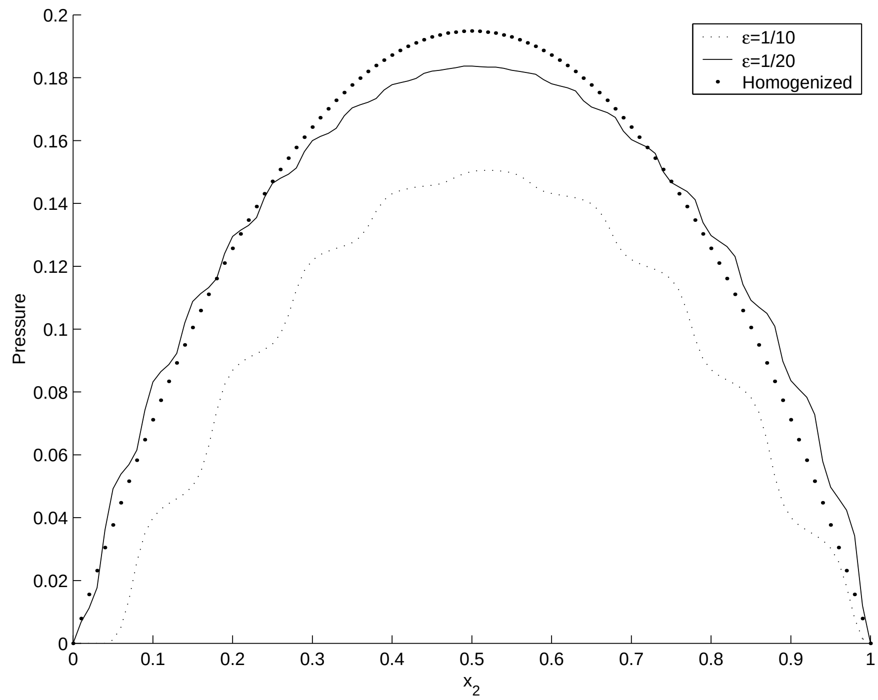


Figure B.3. Hydrodynamic pressure with 2D roughness patterns at $x_1 = 2.639$

of ε as well as the homogenized curves. Due to the number of discretized elements for solving the real problem, it is difficult to compute solutions for values of ε smaller than $1/20$. However, the convergence for the pressure is observed in both directions, and the

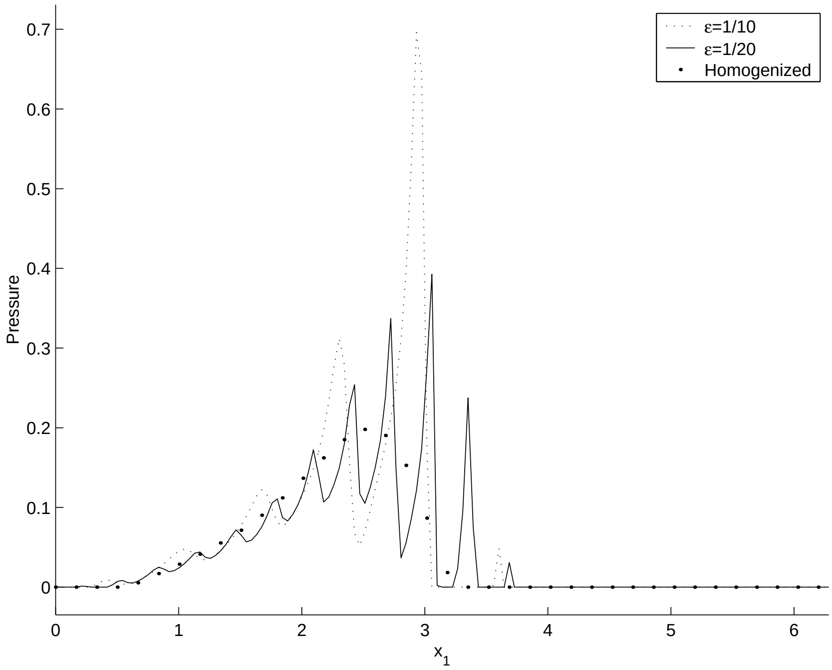


Figure B.4. Hydrodynamic pressure with 2D roughness patterns at $x_2 = 0.5$

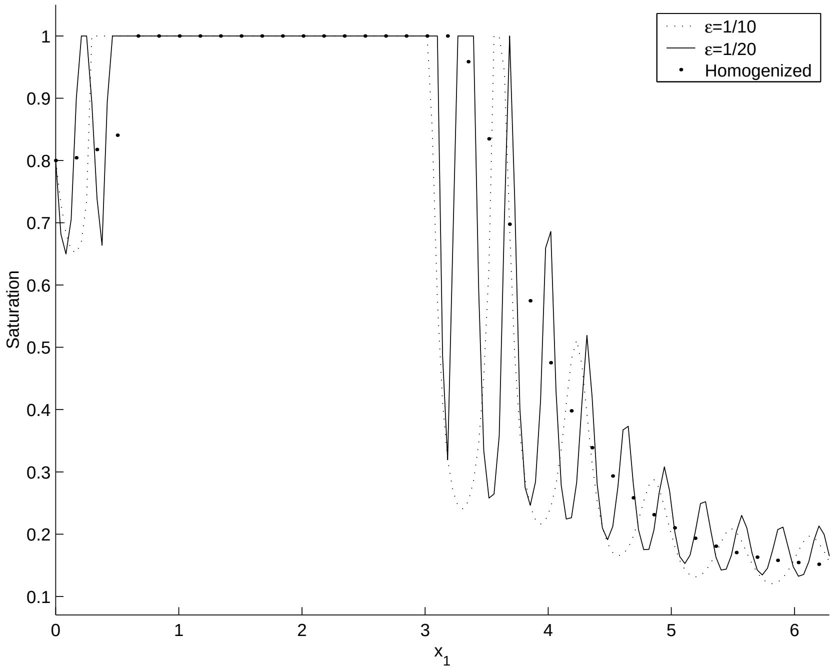


Figure B.5. Hydrodynamic saturation with 2D roughness patterns at $x_2 = 0.5$

same comments as in the transverse roughness case can be made.

B.4.4 Oblique roughness effects

For convenience in computation, the data are not similar to the ones used in the previous subsections: considering the problem given in the real variables (see Equations (B.53)–(B.56)), we choose the following scaling process:

$$x_1 = \frac{X'_1}{2\pi R_m}, \qquad x_2 = \frac{X'_2}{2\pi R_m}, \qquad h_s = \frac{H_s}{c},$$
$$p = \frac{c^2}{6\mu U 2\pi R_m}\, P, \qquad Q = \frac{Q_R}{cU}, \qquad \kappa = \frac{2\pi R_m}{L}.$$

Now, the dimensionless Equations (B.57)–(B.60) are considered with the following data:

- $\kappa = 0.2$, i.e. $2\pi R_m = 0.2L$.
- The domain being $(0,1)\times(0,0.2)$, the rough dimensionless gap is given by:

$$h(x, x/\varepsilon) = h_s(x) + h_r(x/\varepsilon) = 1 + 0.5\ \cos\left(2\pi\frac{e_\gamma\cdot x}{\varepsilon}\right),$$

  with $e_\gamma = (cos\gamma, \sin\gamma)$, $x = (x_1, x_2)$ and $\gamma = \pi/4$, h_s (respectively h_r) denoting the smooth (respectively rough) contribution to the gap.
- The dimensionless flow at the supply groove is $Q = \theta_{in} h_s(0)$ with $\theta_{in} = 0.6$.

FIG.B.6 shows the behaviour of the pressure at a fixed x_2, thus clearly establishing the convergence of the pressure. FIG.B.7 represents the pressure on the supply line ($x_1 = 0$), corresponding to the maximum pressure for the homogenized solution.

FIG.B.8 shows the evolution of the cavitated areas when ε tends to 0. Lubricated (respectively cavitated) zones are coloured in white (respectively black). For not too small values of ε, the direction of the cavitation streamlines is the one of the roughness pattern. This does not seem to be the case for the homogenized one.

The results point out the fact that nondiagonal terms in the left-hand side and extra term in the right-hand side of the homogenized Equation (B.24) or (B.49) are actually needed.

B.4.5 Some remarks on interasperity cavitation

In [HS01], Harp and Salant have proposed an average equation for modelling interasperity cavitation from JFO mass flow preserving model. Basic assumptions are the existence of a (not too small) leading value of the period of the roughness (length of correlation λ) and that the roughness is distributed in a somewhat stochastic way. Then the value of λ does

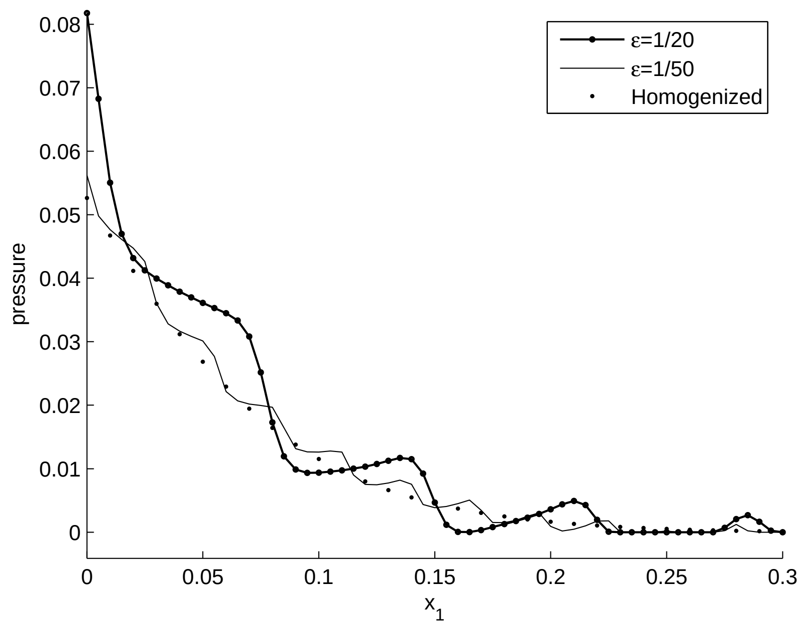


Figure B.6. Hydrodynamic pressure for oblique roughness patterns at $x_2 = 0.1$

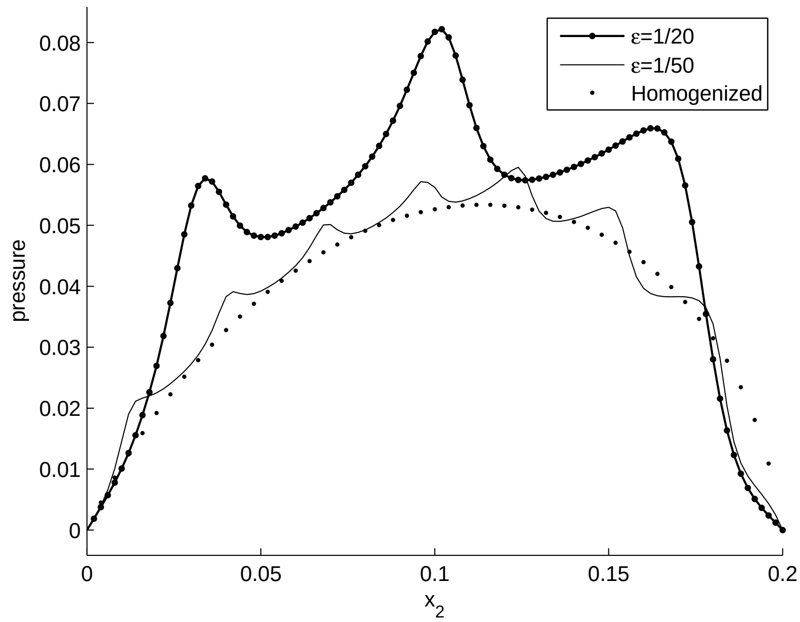


Figure B.7. Hydrodynamic pressure for oblique roughness patterns at $x_1 = 0$

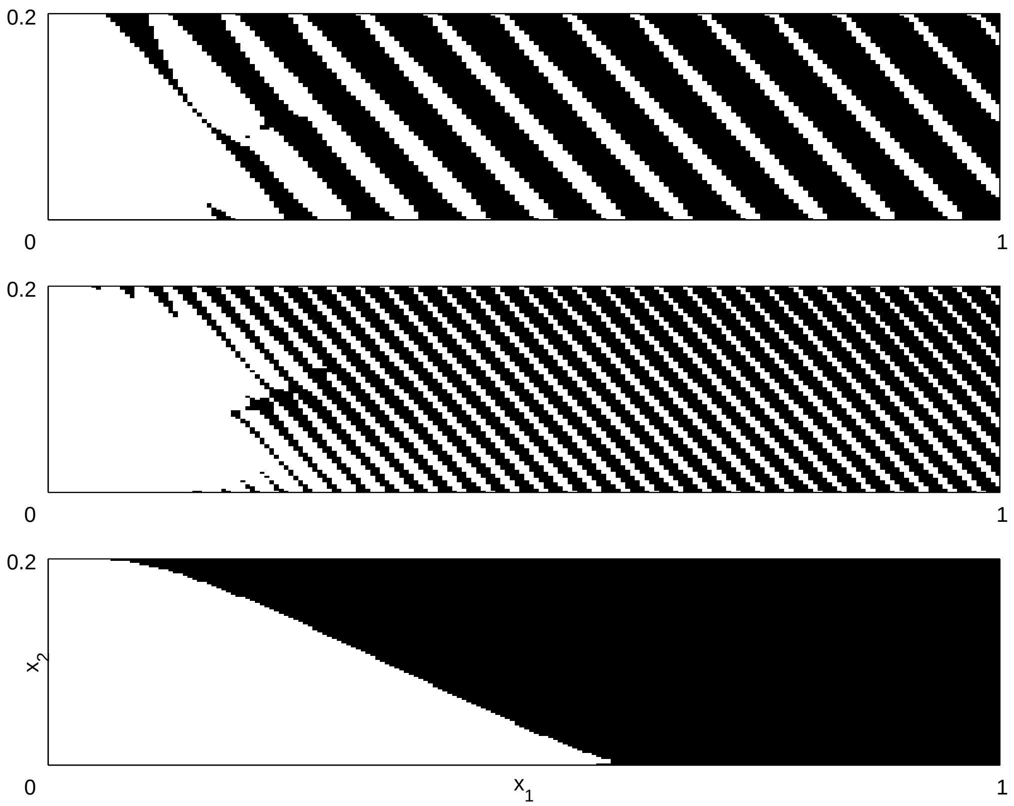


Figure B.8. Lubricated [white] and cavitated [black] areas for different values of ε: 1/20, 1/50, homogenized

not disappear in the average equation obtained in [HS01] and allows for a description in detail of the saturation in the interasperity. However, this equation is questionable for general roughness patterns as neither extradiagonal terms in the left hand-side nor a derivative with respect to the second direction in the right hand-side appear in the average equation, unlike to our present Equation (B.24). This fact has been already pointed out in [BF89] and is directly related to assumptions (4) in [HS01]. In true two dimensional roughness, it is important to take it into account, even without cavitation (see also [BJ04]). For one dimensional roughness as the one numerically studied in [HS01], it is well known that these additional terms no longer exist, so that some comparison can be made between the two approaches.

FIG.B.9 describes numerical results linked to Harp and Salant's comments (in particular Example 2, p. 141 in [HS01]). The data are the following ones:

- The domain is a small square bearing $]0, l[\times]0, l[$ whose area is $l^2 = 0.36\ mm^2$.

- Periodic boundary conditions are placed on $x_1 = 0$ and $x_1 = l$.

- Pressure is imposed on other sides: $p = 1.10^5\ Pa$ on $x_2 = 0$ and $p = 6.10^5\ Pa$ on $x_2 = l$.
- The effective gap is given by:

$$h_\varepsilon(x) = c\left(1 + 0.5\cos\left(\frac{2\pi}{l}\frac{x_1}{\varepsilon}\right)\right)$$

  with $c = 9.10^{-6}\ m$.
- The viscosity is $\mu = 0.2\ N.m.s^{-2}$.
- The velocity is $U = 1\ m.s^{-1}$.

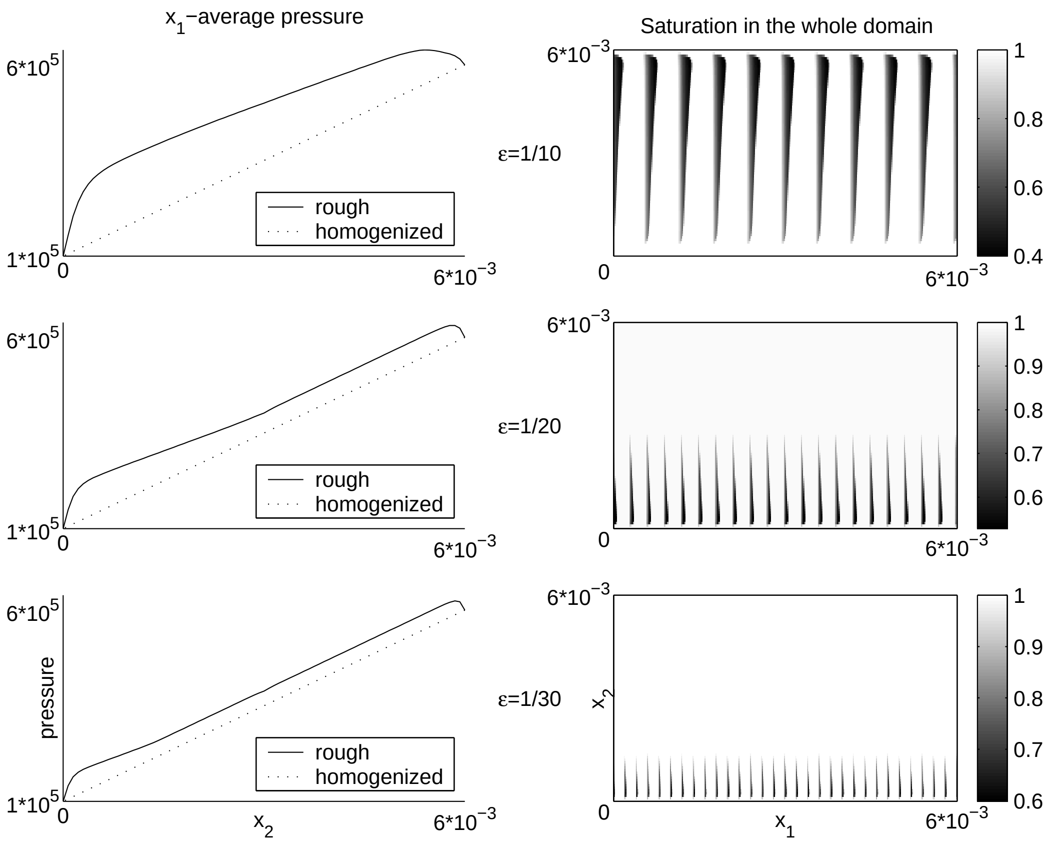


Figure B.9. Average pressure and cavitated areas with interasperity

FIG.B.9 describes on the right-hand side the evolution of the saturation as a function of ε. The related cavitated area consists of a set of elements whose width is thiner with epsilon and whose number is proportional to $1/\varepsilon$. In the homogenized case, the cavitation disappears.

Comparing with the results obtained in [HS01], we can observe that averaging the pressure in the x_1 direction gives the same kind of curves. Moreover, when ε tends to 0, the results are identical with both approaches, as the jump of the pressure at the boundary, introduced in [HS01], decreases with an order ε.

B.5 Conclusion

A solution procedure for deterministic periodic roughness computation has been developped. The procedure uses homogenization multiscale approach and rigorously takes mass flow conservation into account. Classical JFO algorithms can easily be extended to numerically compute the solution of the homogenized Reynolds equation for transverse, longitudinal, oblique and even some two dimensional roughness.

However, further mathematical developments are needed to cope with general two dimensional roughness due to anisotropic effects on the saturation.

Bibliography

[BC86] G. Bayada and M. Chambat. Sur quelques modélisations de la zone de cavitation en lubrification hydrodynamique. *J. Méc. Théor. Appl.*, 5(5):703–729, 1986.

[BC88] G. Bayada and M. Chambat. New models in the theory of the hydrodynamic lubrication of rough surfaces. *ASME J. Tribol.*, 110:402–407, 1988.

[BCV98] G. Bayada, M. Chambat, and C. Vázquez. Characteristics method for the formulation and computation of a free boundary cavitation problem. *J. Comput. Appl. Math.*, 98(2):191–212, 1998.

[BF89] G. Bayada and J.-B Faure. A double-scale analysis approach of the Reynolds roughness. Comments and application to the journal bearing. *ASME J. Tribol.*, 111:323–330, 1989.

[BJ04] G. Buscaglia and M. Jai. Homogenization of the generalized Reynolds equation for ultra-thin gas films and its resolution by FEM. *ASME J. Tribol.*, 126:547–552, 2004.

[BMV05] G. Bayada, S. Martin, and C. Vázquez. Effets d'anisotropie par homogénéisation dans un problème à frontière libre. *C. R. Math. Acad. Sci. Paris*, 340(7):541–546, 2005.

[Bre86] D. E. Brewe. Theoretical modeling of vapor cavitation in dynamically loaded journal bearings. *ASME J. Lub. Technol.*, 108:628–638, 1986.

[CT71] D. G. Christensen and K. Tonder. The hydrodynamic lubrication of rough bearing surfaces of finite width. *ASME J. Lub. Technol.*, 93:324–330, 1971.

[DMT84] D. Dowson, A. A. S. Miranda, and C. Taylor. Implementation of an algorithm enabling the determination of film rupture and reformation boundaries in a film bearing. in *Proceedings of 10th Leeds-Lyon Symposium of Tribology.* Butterworths, U.K., Paper III (ii), 1984.

[EA75] H. G. Elrod and M. L. Adams. A computer program for cavitation. *Cavitation and related phenomena in lubrication - Proceedings - Mech. Eng. Publ. ltd*, pages 37–42, 1975.

[Elr81] H. G. Elrod. A cavitation algorithm. *ASME J. Lubrication Technol.*, 103:350–354, 1981.

[FJ57] L. Floberg and B. Jakobsson. The finite journal bearing considering vaporization. *Transactions of Chalmers University of Technology, Gutenberg, Sweden*, 190, 1957.

[Hoo98] C. J. Hooke. The behaviour of low-amplitude surface roughness under line contacts. *Proc. Instn. Mech. Engrs.*, 213:275–285, 1998.

[HS01] S. H. Harp and R. F. Salant. An average flow model of rough surface lubrication with inter-asperity cavitation. *ASME J. Tribol.*, 123:134–143, 2001.

[JBS02] M. Jai and B. Bou-Saïd. A comparison of homogenization and averaging techniques for the treatment of roughness in slip-flow-modified Reynolds equation. *ASME J. Tribol.*, 124:327–335, 2002.

[KB91] A. Kumar and J. F. Booker. A finite element cavitation algorithm. *ASME J. Tribol.*, 113 (2):276–86, 1991.

[KCNO80] A. L. Kistler, H. S. Cheng, K. Nivatvongs, and I. Ozakat. Cavitation phenomenon in face seals. *ONR Contract N00014-79-0007*, 1980.

[Ols65] K. O. Olsson. Cavitation in dynamically loaded bearing. *Transactions of Chalmers University of Technology, Guthenberg, Sweden*, 308, 1965.

[PC78] N. Patir and H. S. Cheng. An average flow model for determining effects of three-dimensional roughness on partial hydrodynamic lubrication. *ASME J. Lubrication Technol.*, 100:12–17, 1978.

[PS92] P. Payvar and R. F. Salant. A computational method for cavitation in a wavy mechanical seal. *ASME J. Tribol.*, 114:119–204, 1992.

[SS00] F. Shi and R. F. Salant. A mixed soft elastohydrodynamic lubrication model with interasperity cavitation and surface shear deformation. *ASME J. Tribol.*, 122:308–316, 2000.

[SSar] R. B. Siripuram and L. S. Stephens. Effect of deterministic asperity geometry on hydrodynamic lubrication. *ASME J. Tribol.*, 2004 (to appear).

[VK90] D. Vijayaraghavan and T. G. Keith. An efficient, robust, and time accurate numerical scheme applied to a cavitation algorithm. *ASME J. Tribol.*, 112:44–51, 1990.

C

Microroughness effects in EHL lubrication with a mass-flow preserving cavitation model

Article soumis pour publication.

Abstract *An average Reynolds equation is proposed for predicting the effects of deterministic periodic roughness, taking JFO mass flow preserving cavitation model and elastohydrodynamic effects into account. For this, the asymptotic model is based upon double scale analysis approach. The average Reynolds equation can be used both for microscopic interasperity cavitation and macroscopic one. The validity of such a model is verified by numerical experiments.*

NOMENCLATURE

h_r	=	rigid gap
$h_{[p]}$	=	effective gap (including deformation)
h_0	=	minimum thickness of the rigid gap
R	=	sphere or cylinder section radius
k	=	Hertz kernel
p	=	pressure
p_0, p_1...	=	approximations of the pressure
θ	=	saturation
θ_0	=	microscopic homogenized saturation
Θ_i, Θ	=	macro-homogenized saturations
$x = (x_1, x_2)$	=	space variables
$y = (y_1, y_2)$	=	microscale variables
$Y =]0,1[\times]0,1[$	=	rescaled microcell

$(A^{\star}_{[p]})_{ij}$, $A^{(i,\star)}_{[p]}$	=	homogenized coefficients
$B^{(i,\star)}_{[p]}$, $B^{(i,0)}_{[p]}$	=	homogenized coefficients
$w^{(i)}_{[p_0]}$, $\chi^{(i,0)}_{[p_0]}$	=	auxiliary functions defined on Y
α	=	piezoviscosity coefficient
$\partial/\partial n$	=	normal derivative
ε	=	roughness spacing
$\overline{\cdot}^{Y}$	=	average operator with respect to y
L	=	length of the journal bearing
R_b	=	bearing radius
R_j	=	journal radius
R_m	=	mean radius
Γ_a, Γ_0, $\Gamma_{\sharp}$, $\Gamma_{\star}$	=	boundaries of the device
c	=	clearance
ρ	=	eccentricity of the bearing
p_a	=	supply pressure
μ	=	viscosity
v_0	=	velocity of the bearing
a_r	=	amplitude of the roughness
W	=	load
θ_{in}	=	supply flow

C.0 Introduction

In this paper, it is explained how the double scale procedure, already used to obtain average equations with periodic roughness in the case of rigid bearings [BF89, BMV05b, BMV05a], can be extended to EHD problems including cavitation and starvation. The JFO mass flow preserving model is used, including pressure and saturation as unknown functions. This model takes into account both microcavitation (due to the microroughness) and macrocavitation (due to the diverging part of the gap). Average equation can be easily solved for some specific roughness patterns (transverse, longitudinal) exactly in the same way as the initial EHD problem with cavitation. Numerical results are given for both purely hydrodynamic and EHD point-contact problems, for a two dimensional device.

C.1 Basic equations

Our studied cavitation model, like the Elrod algorithm and its variants [KB91, SS00], views the film as a mixture. It does not, however, make the assumption of liquid compressibility in the full film area as in [VK90] and some other papers. The flow obeys the following "universal" Reynolds equation (here written in a dimensionless form) through all the gap in which the pressure cavitation is assumed to be zero in the cavitation area. Moreover, the lubricant is piezoviscous so that the viscosity obeys the Barus law (notice that other laws may be taken into account). The effective gap contains a rigid contribution and an elastic one, which is given by the Hertz law (for local contacts):

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(h_{[p]}^3 \, e^{-\alpha p} \, \frac{\partial p}{\partial x_i} \right) = \frac{\partial \, \theta h_{[p]}}{\partial x_1}, \tag{C.1}$$

$$p \geq 0, \tag{C.2}$$

$$0 \leq \theta \leq 1, \tag{C.3}$$

$$p \, (1 - \theta) = 0, \tag{C.4}$$

p is the pressure (assumed to be a positive function), θ is the relative mixture density, $h_{[p]}$ the real film thickness, x_1 is the direction of the effective relative shear velocity of the device, while x_2 is the transverse direction. Here, $h_{[p]}$, which is the effective gap between two close surfaces, contains a given rigid contribution h_r and an elastic one, which strongly depends on the main unknown p (lubricant pressure) in the following nonlocal form:

$$h_{[p]}(x) = h_r(x) + \int_\Omega k(x, z) p(z) \; dz,$$

the kernel k depending on the kind of contact. The classical approximation of the rigid gap [DH77] is given by the expression

$$h_r(x) = \begin{cases} h_0 + \dfrac{x_1^2 + x_2^2}{2R}, & \text{for ball bearings,} \\ h_0 + \dfrac{x_1^2}{R}, & \text{for linear bearings,} \end{cases} \tag{C.5}$$

that represents a parabolic approximation for a given sphere-plane (point contact) or cylinder-plane (line contact) gap, R being the sphere or cylinder section radius. The positive constant h_0 corresponds to the gap at the point nearest to contact. Now, let us introduce the general property of the kernel k:

$$k(x,z) = \begin{cases} c_0 \ \log\left|\dfrac{c_1 - z_1}{x_1 - z_1}\right|, & \text{for line contacts,} \\ \dfrac{c_0}{\sqrt{(x_1-z_1)^2 + (x_2-z_2)^2}}, & \text{for point contacts,} \end{cases} \tag{C.6}$$

where $c_0 > 0$ and $c_1 \geq \max\{\left|x_1\right|,\ x \in \overline{\Omega}\}$.

C.2 Asymptotic expansion

Let us suppose that the roughness is periodically reproduced in the two x_1 and x_2 directions from an elementary cell Y (or "miniature bearing" in Tonder's terminology). We denote by ε the ratio of the homothetic transformation passing from the elementary cell $Y = Y_1 \times Y_2$ to the real bearing and by $y_1 = x_1/\varepsilon$ and $y_2 = x_2/\varepsilon$ the local variables (see Fig. B.1 in Appendix B).

Let us now consider gaps that can be written as $h_r(x, x/\varepsilon)$. Introducing now the fast variables y_1 and y_2, it appears that the new expression for the gap is $h_r(x,y)$ in the two-scale approach. The combined computation in terms of (x_1, x_2) or (y_1, y_2) is an important feature of the method. It is convenient to consider first x and y as independent variables and to replace next y by x/ε (see [BF89]).

In this section, we recall the main tools that allow the derivation of average equations. The process has the same feature as the one corresponding to the hydrodynamic case, which can be found in [BMV05a]. The equations should be read as

$$\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left\{\left(h_{[p]}\left(x, \frac{x}{\varepsilon}\right)\right)^3 e^{-\alpha p}\frac{\partial p}{\partial x_j}\right\} = \frac{\partial}{\partial x_1}\left\{\theta\ h_{[p]}\left(x, \frac{x}{\varepsilon}\right)\right\},$$
$$p \geq 0, \quad 0 \leq \theta \leq 1, \quad p(1-\theta) = 0,$$

with the additional notation:

$$h_{[p]}\left(x, \frac{x}{\varepsilon}\right) = h_r\left(x, \frac{x}{\varepsilon}\right) + \int_\Omega k(x,z)\ p(z)\ dz.$$

The underscript ε indicates the dependance of the real pressure / saturation on the microtexture related to ε. We shall look for an asymptotic expansion of the solutions:

$$p(x) = p_0(x) + \varepsilon p_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 p_2\left(x, \frac{x}{\varepsilon}\right) + ..., \tag{C.7}$$

$$\theta(x) = \theta_0\left(x, \frac{x}{\varepsilon}\right), \tag{C.8}$$

p_0 being a positive function, and each unknown p_i $(i \geq 1)$ and θ_0 being function of (x, y). The problem of the boundary conditions to be satisfied by the p_i is somewhat difficult but may be summarized so:

(i) The natural boundary conditions on (p, θ) are assigned to p_0 and an equivalent saturation linked to θ_0, which will be developped in next subsection.

(ii) The function p_i, $i \geq 1$, are Y periodic, i.e. periodic in the two variables y_1, y_2, for each value of (x_1, x_2).

To be noticed that, unlike of p, we do not introduce an asymptotic expansion for θ. This can be explained by observing the evolution of p and θ as ε tends to 0. Clearly, the oscillations of the pressure are decreasing and p tends to a smooth function (namely p_0 which, actually, does not depend on the fast variable as it will be pointed out further). This is not the case for θ and an asymptotic smooth limit cannot be considered.

We shall see later that the functions p_i, $i \geq 1$, are defined up to an additive constant. Moreover, from Equations (C.2)–(C.4), the following properties hold:

$$p_0(x) \geq 0, \tag{C.9}$$

$$0 \leq \theta_0(x, y) \leq 1, \tag{C.10}$$

$$p_0(x, y)\ (1 - \theta_0(x, y)) = 0. \tag{C.11}$$

Putting Equations (C.7) and (C.8) into Equation (C.1), one can write, by an identification procedure, the following macroscopic / microscopic decomposition:

- Macroscopic equation:

$$\begin{aligned} &\sum_{j=1}^{2} \frac{\partial}{\partial x_j} \left\{ \overline{h^3_{[p_0]}(x, y)\ e^{-\alpha p_0(x)}\ \left(\frac{\partial p_0}{\partial x_j}(x) + \frac{\partial p_1}{\partial y_j}(x, y) \right)}^{Y} \right\} \\ &\qquad = \frac{\partial}{\partial x_1} \left\{ \overline{\theta_0(x, y)\ h_{[p_0]}(x, y)}^{Y} \right\}, \end{aligned} \tag{C.12}$$

where $\overline{u}^Y$ denotes the local average of any Y periodic function u, i.e. $\overline{u}^Y(x) = \dfrac{1}{[Y]} \displaystyle\int_Y u(x, y)\, dy$.

- Microscopic equation: for each x, we have

$$\begin{aligned} &\sum_{j=1}^{2} \frac{\partial}{\partial y_j} \left\{ h^3_{[p_0]}(x, y)\ e^{-\alpha p_0(x)}\ \left(\frac{\partial p_0}{\partial x_j}(x) + \frac{\partial p_1}{\partial y_j}(x, y) \right) \right\} \\ &\qquad = \frac{\partial}{\partial x_1} \Big\{ \theta_0(x, y)\ h_{[p_0]}(x, y) \Big\}. \end{aligned} \tag{C.13}$$

Notice that, in the earlier equation, the variable x only plays the role of a parameter.

Of course, the macroscopic equation (C.12) is not sufficient to describe a complete asymptotic model: indeed, we have to eliminate p_1 in order to get a single pressure-saturation equation. For this, we use the microscopic equation (C.13). Thus we can represent p_1 as a function of p_0 and θ_0 in a more usable form. We define $w^{(i)}_{[p_0]}$, $\chi^{(i),0}_{[p_0]}$ and $\chi^{(i,\star)}_{[p_0]}$ $(i = 1, 2)$ as the Y periodic solutions (up to an additive constant) of the following local problems:

$$\sum_{j=1}^{2} \frac{\partial}{\partial y_j} \left(h^3_{[p_0]} \; e^{-\alpha p_0} \; \frac{\partial w^{(i)}_{[p_0]}}{\partial y_j} \right) = \frac{\partial h^3_{[p_0]}}{\partial y_i}, \tag{C.14}$$

$$\sum_{j=1}^{2} \frac{\partial}{\partial y_j} \left(h^3_{[p_0]} \; e^{-\alpha p_0} \; \frac{\partial \chi^{(i,0)}_{[p_0]}}{\partial y_j} \right) = \frac{\partial \theta_0 h_{[p_0]}}{\partial y_i}, \tag{C.15}$$

$$\sum_{j=1}^{2} \frac{\partial}{\partial y_j} \left(h^3_{[p_0]} \; e^{-\alpha p_0} \; \frac{\partial \chi^{(i,\star)}_{[p_0]}}{\partial y_j} \right) = \frac{\partial h_{[p_0]}}{\partial y_i}. \tag{C.16}$$

Notice that, contrary to the local problems in the purely hydrodynamic case, the local problems in the EHL problem highly depend on the macroscopic pressure p_0, which is due to the piezoviscosity and elastic deformation only.

The solution of Equation (C.13) reduces to:

$$\begin{aligned} p_1(x,y) = \chi^{(1,0)}_{[p_0]}(x,y) \quad &- \quad \frac{\partial p_0}{\partial x_1}(x) \; w^{(1)}_{[p_0]}(x,y) \\ &- \quad \frac{\partial p_0}{\partial x_2}(x) \; w^{(2)}_{[p_0]}(x,y). \end{aligned} \tag{C.17}$$

By exchanging the integral and the derivation symbols, and after some calculations, Equation (C.12) becomes:

$$\sum_{i,j} \frac{\partial}{\partial x_i} \left\{ \left(A^{\star}_{[p_0]} \right)_{ij} e^{-\alpha p_0} \; \frac{\partial p_0}{\partial x_j} \right\} = \sum_i \frac{\partial}{\partial x_i} \left\{ B^{(i,0)}_{[p_0]} \right\},$$

where $(i, j = 1, 2 \text{ and } j \neq i)$

$$\begin{aligned}
\left(A^{\star}_{[p_0]}\right)_{ii} &= \overline{h^3_{[p_0]} - h^3_{[p_0]} \frac{\partial w^{(i)}_{[p_0]}}{\partial y_i}}^{Y}, \\
\left(A^{\star}_{[p_0]}\right)_{ij} &= \overline{-h^3_{[p_0]} \frac{\partial w^{(j)}_{[p_0]}}{\partial y_i}}^{Y} = \overline{-h^3_{[p_0]} \frac{\partial w^{(i)}_{[p_0]}}{\partial y_j}}^{Y} = \left(A^{\star}_{[p_0]}\right)_{ji},
\end{aligned}$$

and also

$$B^{(1,0)}_{[p_0]} = \overline{\theta_0 h_{[p_0]} - h^3_{[p_0]} \frac{\partial \chi^{(1,0)}_{[p_0]}}{\partial y_1}}^{Y}, \quad B^{(2,0)}_{[p_0]} = \overline{-h^3_{[p_0]} \frac{\partial \chi^{(1,0)}_{[p_0]}}{\partial y_2}}^{Y}.$$

In the end, we can also define the normalized coefficients

$$\begin{aligned}
B^{(1,\star)}_{[p_0]} &= \overline{h_{[p_0]} - h^3_{[p_0]} \frac{\partial \chi^{(1,\star)}_{[p_0]}}{\partial y_1}}^{Y}, \qquad & B^{(2,\star)}_{[p_0]} &= \overline{-h^3_{[p_0]} \frac{\partial \chi^{(1,\star)}_{[p_0]}}{\partial y_2}}^{Y}, \\
\Theta_1 &= \frac{B^{(1,0)}_{[p_0]}}{B^{(1,\star)}_{[p_0]}}, & \Theta_2 &= \frac{B^{(2,0)}_{[p_0]}}{B^{(2,\star)}_{[p_0]}},
\end{aligned}$$

which provide the following generalized Reynolds equation:

$$\text{(C.18)} \qquad \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \left\{ \left(A^{\star}_{[p_0]}\right)_{ij} e^{-\alpha p_0} \frac{\partial p_0}{\partial x_j} \right\} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left\{ \Theta_i B^{(i,\star)}_{[p_0]} \right\}.$$

Moreover it is obvious that the following properties hold:

$$\text{(C.19)} \qquad p_0 \geq 0, \quad p_0(1 - \Theta_i) = 0, \quad (i = 1, 2).$$

Equations (C.18) and (C.19) deal with any periodic roughness pattern. To be noticed is the fact that the differential operator is no more of the Reynolds type since extra terms $\partial^2 p_0 / \partial x_i \partial x_j$ appear. The right-hand side also contains an additive term in the x_2 direction. However, the link between p_0 and Θ_i is not so clear. This is a major obstacle which prevents to get a tractable equation. In fact, the pressure / double saturation problem lacks some properties: we cannot prove that the so-called anisotropic saturations Θ_i are functions with values in $[0, 1]$. However, it is possible to define in a rigorous way (see [BMV05b]) a solution with isotropic solution $\Theta = \Theta_1 = \Theta_2$ and $0 \leq \Theta_i \leq 1$. Moreover, under some additional assumptions (see [BMV05a]), we can prove some additional results. Thus, when dealing with transverse and longitudinal roughness, the homogenized coefficients can be easily simplified and given in an explicit form. In both cases, the asymptotic system has the same structure than the initial roughless one, i.e.

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left\{ A_{[p_0]}^{(i,\star)} \; e^{-\alpha p_0} \frac{\partial p_0}{\partial x_i} \right\} = \frac{\partial \Theta B_{[p_0]}^{(1,\star)}}{\partial x_1}, \tag{C.20}$$

$$p_0 \geq 0, \tag{C.21}$$

$$0 \leq \Theta \leq 1, \tag{C.22}$$

$$p_0 \; (1 - \Theta) = 0. \tag{C.23}$$

▷ *Transverse roughness*: the homogenized coefficients are

$$A_{[p_0]}^{(1,\star)} = \frac{1}{\overline{h_{[p_0]}^{-3}}^{Y}}, \quad A_{[p_0]}^{(2,\star)} = \overline{h_{[p_0]}^{3}}^{Y}, \quad B_{[p_0]}^{(1,\star)} = \frac{\overline{h_{[p_0]}^{-2}}^{Y}}{\overline{h_{[p_0]}^{-3}}^{Y}},$$

the link between the macroscopic saturation Θ and the microscopic one θ_0 being given by:

$$\Theta(x) = \left(\frac{1}{\overline{h_{[p_0]}^{-2}}^{Y}} \; \overline{\left(\frac{\theta_0}{h_{[p_0]}^{2}} \right)}^{Y} \right)(x).$$

▷ *Longitudinal roughness*: the homogenized coefficients are

$$A_{[p_0]}^{(1,\star)} = \overline{h_{[p_0]}^{3}}^{Y}, \quad A_{[p_0]}^{(2,\star)} = \frac{1}{\overline{h_{[p_0]}^{-3}}^{Y}}, \quad B_{[p_0]}^{(1,\star)} = \overline{h_{[p_0]}}^{Y},$$

the link between the macroscopic saturation Θ and the microscopic one θ_0 being given by:

$$\Theta(x) = \left(\frac{\overline{\theta_0 h_{[p_0]}}^{Y}}{\overline{h_{[p_0]}}^{Y}} \right)(x).$$

All the earlier results are valid for both elastohydrodynamic and hydrodynamic cases and, thus, generalize the ones that have been stated in [BMV05a]. As an important feature, Θ is not the average of the microscopic saturation θ_0 but contains some anisotropic effects due to the roughness direction. In the purely hydrodynamic case, one can prove some additional results, corresponding to a wide class of two dimensional roughness patterns. Indeed, suppose that h_r can be written under the form

$$h_r(x, y) = h_1(x, y_1) \, h_2(x, y_2),$$

then we get the following (hydrodynamic) homogenized equation

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left\{ A_i^{\star} \frac{\partial p_0}{\partial x_i} \right\} = \frac{\partial}{\partial x_1} \left\{ \Theta B^{(1,\star)} \right\},$$
$$p_0 \geq 0, \quad 0 \leq \Theta \leq 1, \quad p_0(1-\Theta) = 0,$$

with

$$A_i^{\star} = \frac{\overline{h_j^3}^Y}{\overline{h_i^{-3}}^Y}, \quad B^{(1,\star)} = \frac{\overline{h_1^{-2}}^Y}{\overline{h_1^{-3}}^Y} \, \overline{h_2}^Y,$$

the link between the macroscopic and the microscopic saturations being given by

$$\Theta = \frac{1}{\overline{h_2}^Y \, \overline{h_1^{-2}}^Y} \overline{\left(\frac{\theta_0 h_2}{h_1^2} \right)}^Y.$$

C.3 Numerical results

In this section, the numerical simulation of a micro(elasto)hydrodynamic contact is performed to illustrate the theoretical results of convergence stated in the previous sections. For the numerical solution of the ε dependent problems and their corresponding homogenized one, we propose the characteristics method adapted to steady state problems to deal with the the convection term combined with a finite element spatial discretization. These numerical techniques have been already successfully used in previous papers dealing with hydrodynamic aspects (see [BCV98, BD87]), and elastohydrodynamic aspects (see, for instance, [ACV02, DGV02]).

C.3.1 Hydrodynamic case

We adress the numerical simulation of journal bearing devices with circumferential supply of lubricant. The mechanical characteristics of the device are given by:

- length: $L = 0.019\ m$,
- bearing radius: $R_b = 0.0164975\ m$,
- journal radius: $R_j = 0.01647\ m$,
- mean radius: $R_m = 0.5\ (R_b + R_j)$,
- clearance : $c = R_b - R_j$,
- eccentricity: $\rho = 0.2$.

The physical characteristics of the regime are the following one:

- supply pressure: $p_a = 283000\ Pa$,
- lubricant viscosity: $\mu = 0.02\ Pa.s$,
- shear velocity: $v_0 = 17.247\ m/s$.

The earlier problem leads to the following set of equations

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(h^3\ \frac{\partial p}{\partial x_i}\right) = \Lambda \frac{\partial\ \theta h}{\partial x_1},$$
$$p \geq 0, \quad 0 \leq \theta \leq 1, \quad p\ (1-\theta) = 0,$$

with $\Lambda = 6\mu v_0$, and the real roughless gap shoud be read as

$$h(x) = c\,(1 + \rho\ \cos(x_1/R_m))\,.$$

The equations have to be solved on the domain $(0, 2\pi R_m) \times (0, L/2)$ (see the developped configuration on FIG.C.1) with the following boundary conditions:

- $p = p_a$ on Γ_a, $p = 0$ on Γ_0,
- periodic conditions on $\Gamma_\sharp$, i.e. p and $6\mu v_0\ \theta h - h^3 \dfrac{\partial p}{\partial x_1}$ are $2\pi R_m x_1$ periodic.

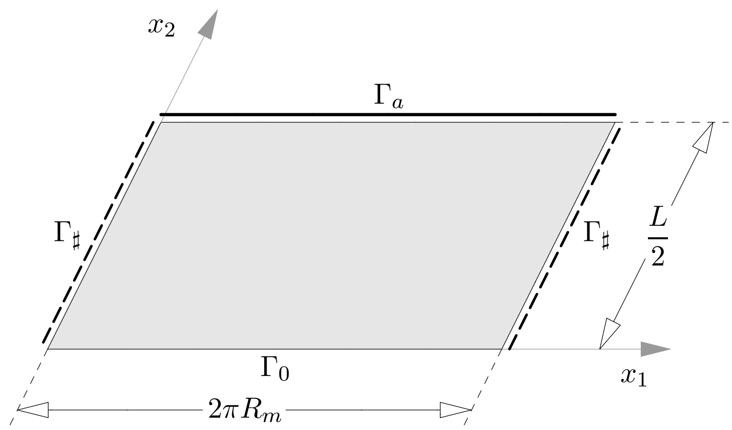


Figure C.1. Journal bearing domain

Actually, we consider transverse roughness patterns and the gap should be read as:

$$h\left(x, \frac{x}{\varepsilon}\right) = c\left(1 + \rho\ \cos(x_1/R_m) + a_r\ \sin\left(\frac{2\pi}{\varepsilon}\frac{x_1}{R_m}\right)\right)$$

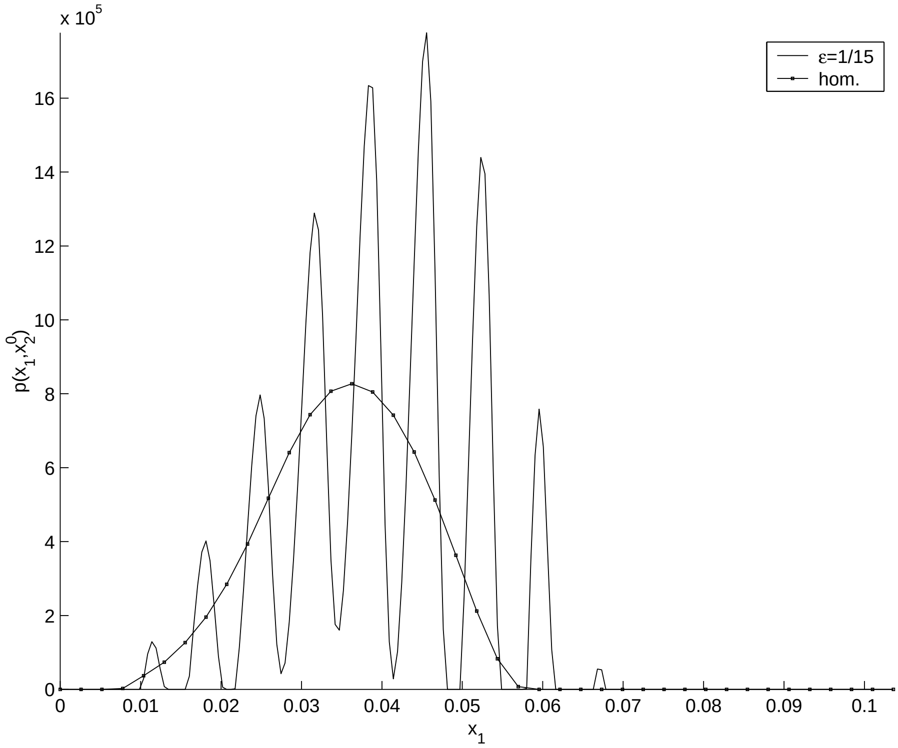


Figure C.2. Hydrodynamic pressure for $\varepsilon = 1/15$ at $x_2^0 = L/4$

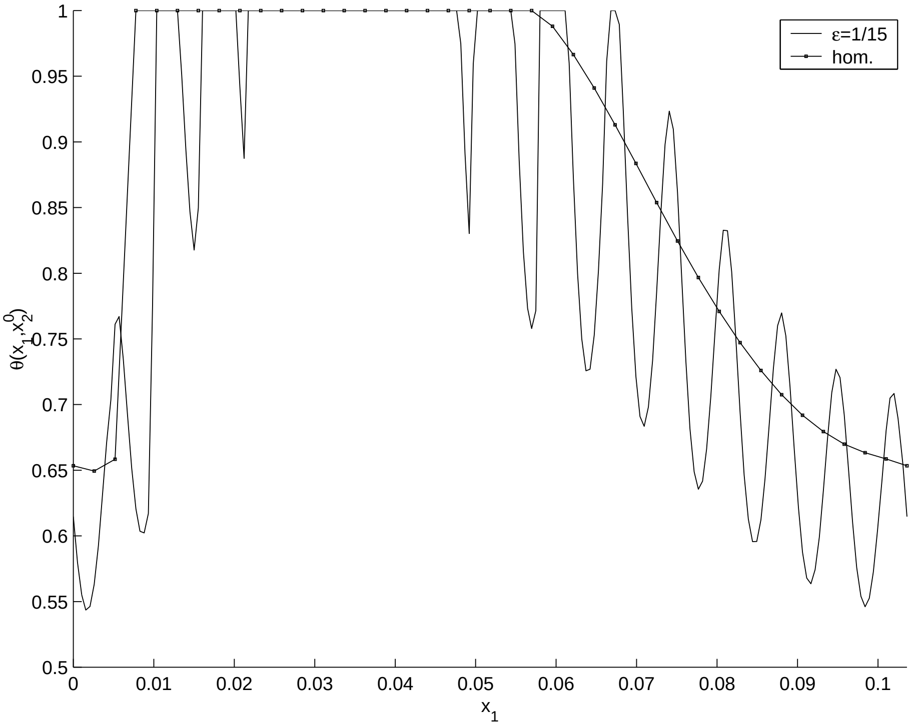


Figure C.3. Hydrodynamic saturation for $\varepsilon = 1/15$ at $x_2^0 = L/4$

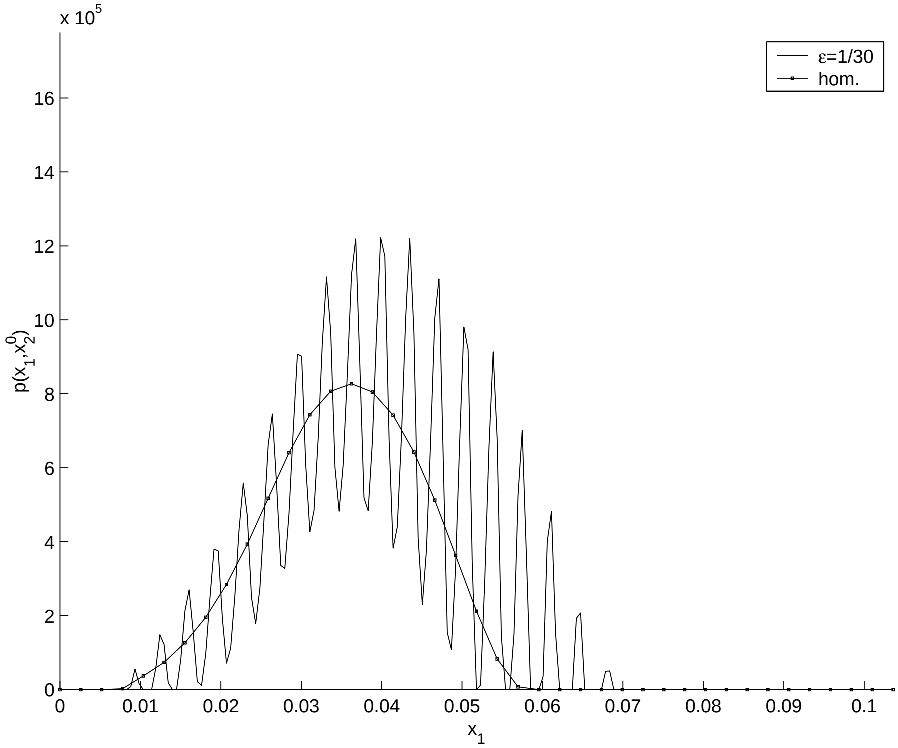


Figure C.4. Hydrodynamic pressure for $\varepsilon = 1/30$ at $x_2^0 = L/4$

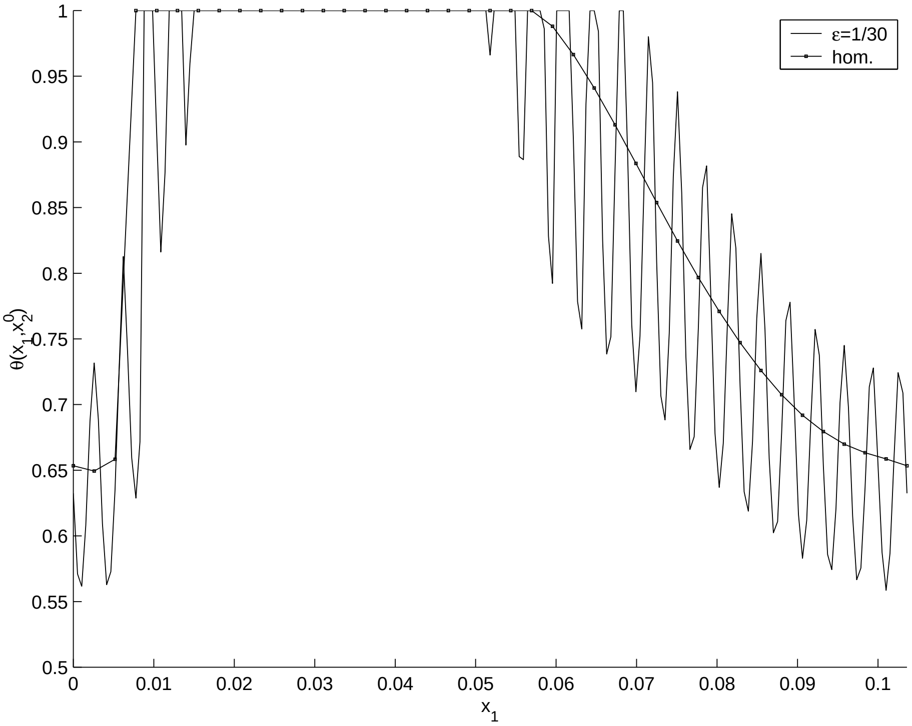


Figure C.5. Hydrodynamic saturation for $\varepsilon = 1/30$ at $x_2^0 = L/4$

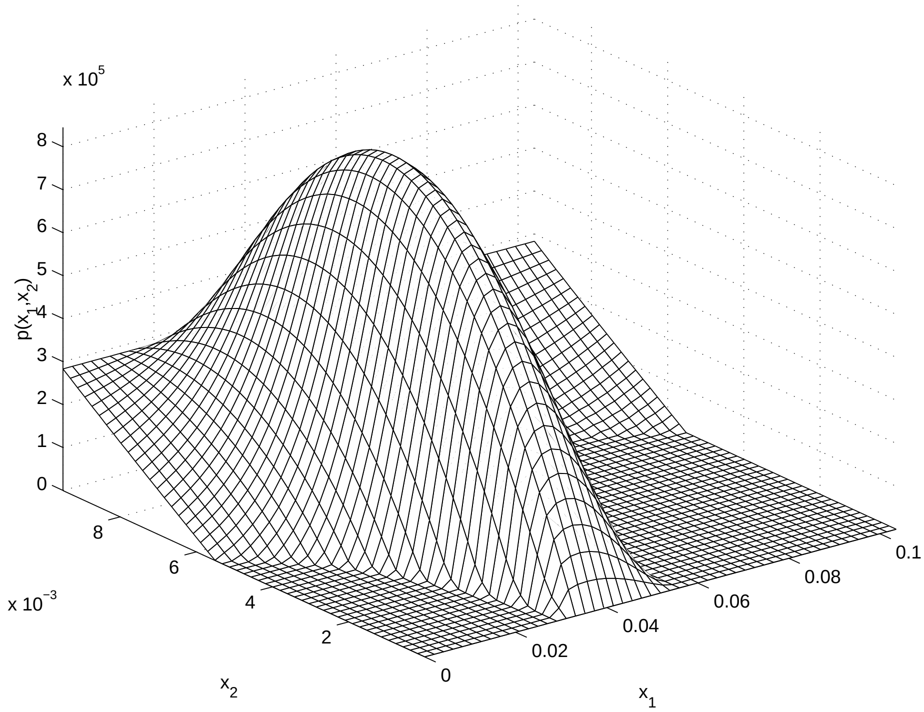


Figure C.6. Homogenized hydrodynamic pressure in the whole device

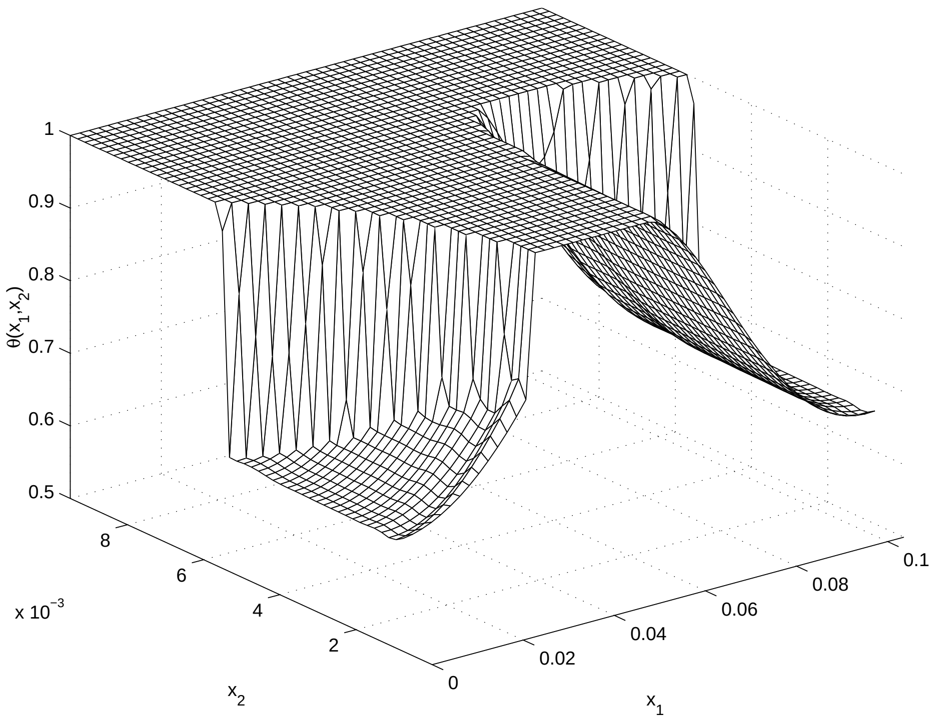


Figure C.7. Homogenized hydrodynamic saturation in the whole device with $a_r/\rho = 0.2$, a_r denoting the amplitude parameter of the roughness. Homogenized coeeficients are deduced from TABLE 1.1 (see page 95).

FIG.C.2 and C.3 (resp. C.4 and C.5) show the pressure and saturation profiles for $\varepsilon = 1/15$ (resp. $\varepsilon = 1/30$) compared to the homogenized solution, at a fixed $x_2^0 = L/4$. Thus, it allows to observe the roughness effects in the x_1 direction. The amplitude of the pressure oscillations tend to be damped, although the amplitude of the saturation oscillations stay the same in cavitated areas. As it was noticed in [BMV05a], it points out the fact that when the number of roughness patterns increases, the pressure behaves as a smooth function, namely $p_0(x)$, while the saturation behaves as a highly oscillating function, namely $\theta_0(x, x/\varepsilon)$. Thus, the pressure tends to a smooth one as ε tends to 0, while the saturation is always oscillating. To be noticed on FIG.C.3 and C.5 is the fact that the cavitation area is made of two macrocavitation zones (for $\varepsilon = 1/15$, $x_1 > 0.06$ and $x_1 < 0.01$) and a lot of microcavitation zones.

FIG.C.6 and C.7 represent the homogenized pressure and saturation in the real domain.

C.3.2 Elastohydrodyamic case

The numerical tests deal with a dimensionless problem, as described in Section C.1: the domain is $(-4, 2) \times (2, 2)$. The considered rigid contribution to the gap is a normalized one:

$$h_0 + \frac{x_1^2 + x_2^2}{2}$$

where h_0 denotes the minimum thickness. Since a point contact has been considered, we choose the following Hertz model:

$$k(x, z) = \frac{2}{\pi^2} \frac{1}{\sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2}}$$

The equations are:

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(h[p]^3 \; e^{-\alpha p} \; \frac{\partial p}{\partial x_i} \right) = \frac{\partial \; \theta h[p]}{\partial x_1},$$
$$p \geq 0, \quad p \; (1 - \theta) = 0 \quad 0 \leq \theta \leq 1,$$

The boundary conditions are the following ones:

- flow condition on $\Gamma_\star$: $\theta h[p] - h[p]^3 \nabla p \cdot n = \theta_{in} h[p]$, with $\theta_{in} = 0.3$,
- $p = 0$ elsewhere.

The chosen values of h_0 and α will be discussed further.

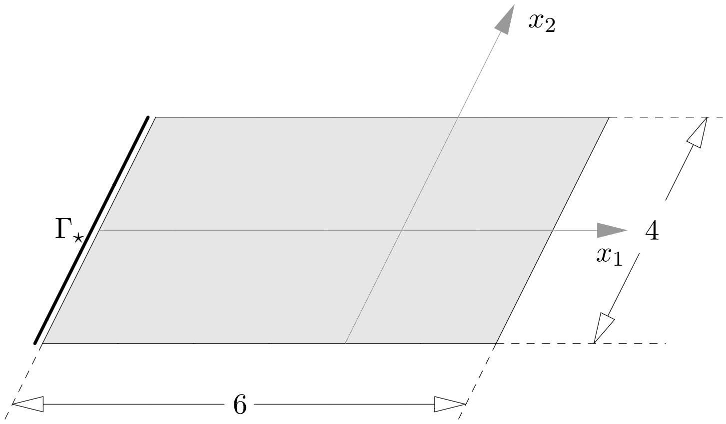


Figure C.8. Normalized EHL domain

Transverse roughness

Numerical tests have been made for the following rigid contribution to the gap:

$$h_0 + \frac{x_1^2 + x_2^2}{2} + h_0 \ \sin\left(2\pi \frac{x_1 + 4}{6\ \varepsilon}\right)$$

with $h_0 = 0.5$ and different values of ε. Moreover, the piezoviscosity has been taken to $\alpha = 1$. Homogenized coefficients in the transverse roughness cases are deduced from TABLE 2.1 (see page 134).

FIG.C.9, C.11 and C.13 represent the pressure, saturation and deformation profiles at $x_2^0 = 0$ in the roughless case ($\varepsilon = +\infty$), a deterministic rough case ($\varepsilon = 1/30$) and the homogenized case. To be observed is the fact that the homogenized profiles give a satisfying approach of the roughness effects ($\varepsilon = 1/30$), unlike the roughless profiles: indeed, the pressure profiles given in FIG.C.9 evidence the fact that the homogenized pressure is a smooth version of the rough pressure. Similarly, by FIG.C.11, the homogenized saturation can be seen as an average version of the rough saturation, up to anisotropic effects. On FIG.C.13, we observe that the rough deformation (corresponding to $\varepsilon = 1/30$) nearly coincides with the homogenized one: this is due to the regularizing effects of the Hertz kernel. In fact, the deformation profile has a rate of convergence which is much greater than the pressure profile.

FIG.C.10, C.12 and C.14 represent the homogenized pressure, saturation and deformation in the domain.

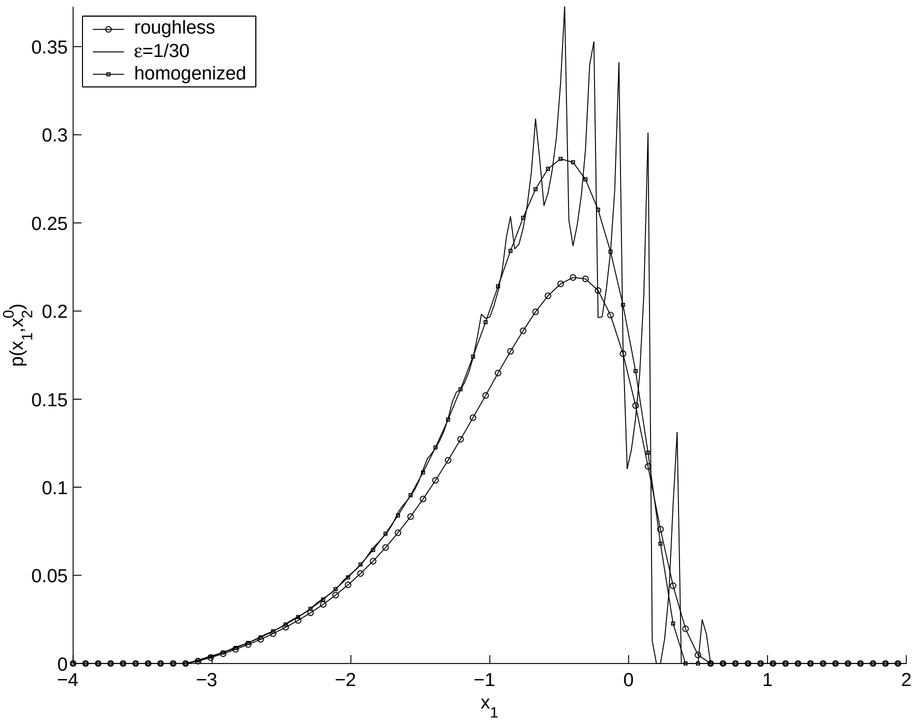


Figure C.9. EHL pressure with transverse roughness patterns at $x_2^0 = 0$

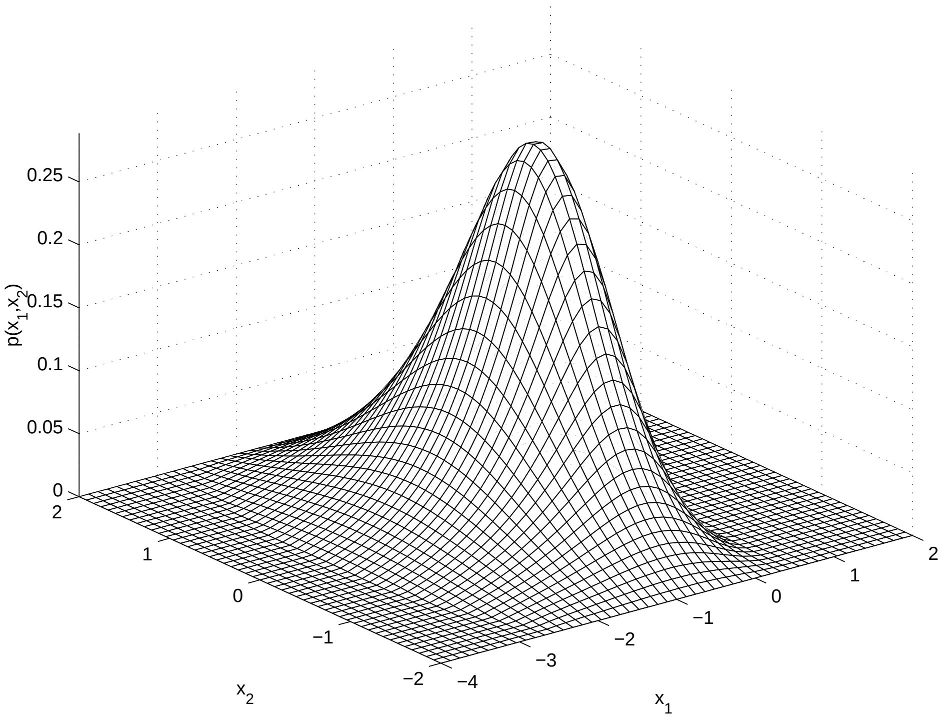


Figure C.10. Homogenized EHL pressure in the whole domain

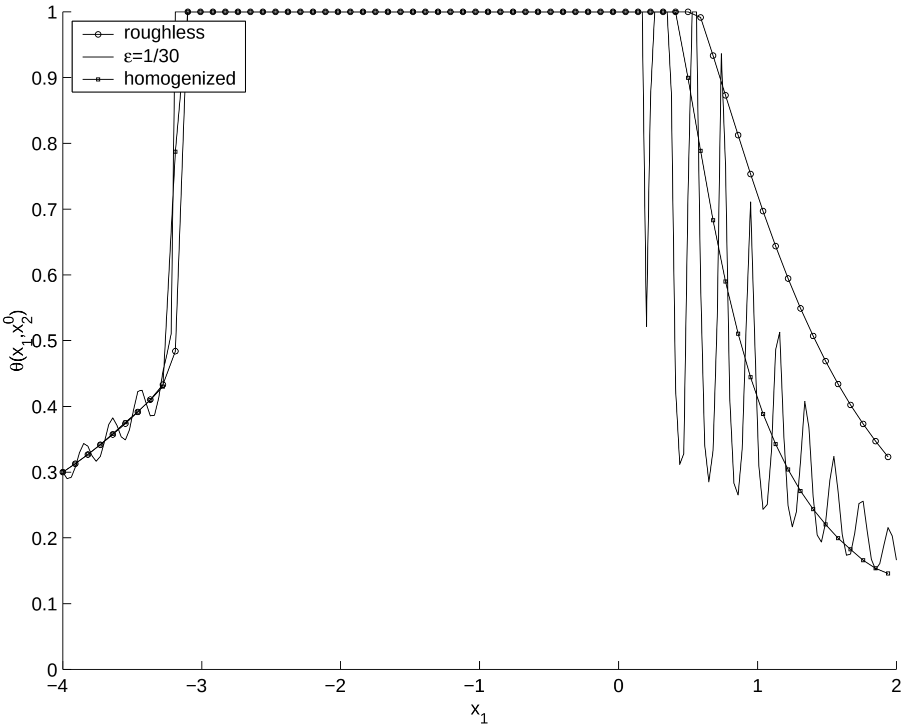


Figure C.11. EHL saturation with transverse roughness patterns at $x_2^0 = 0$

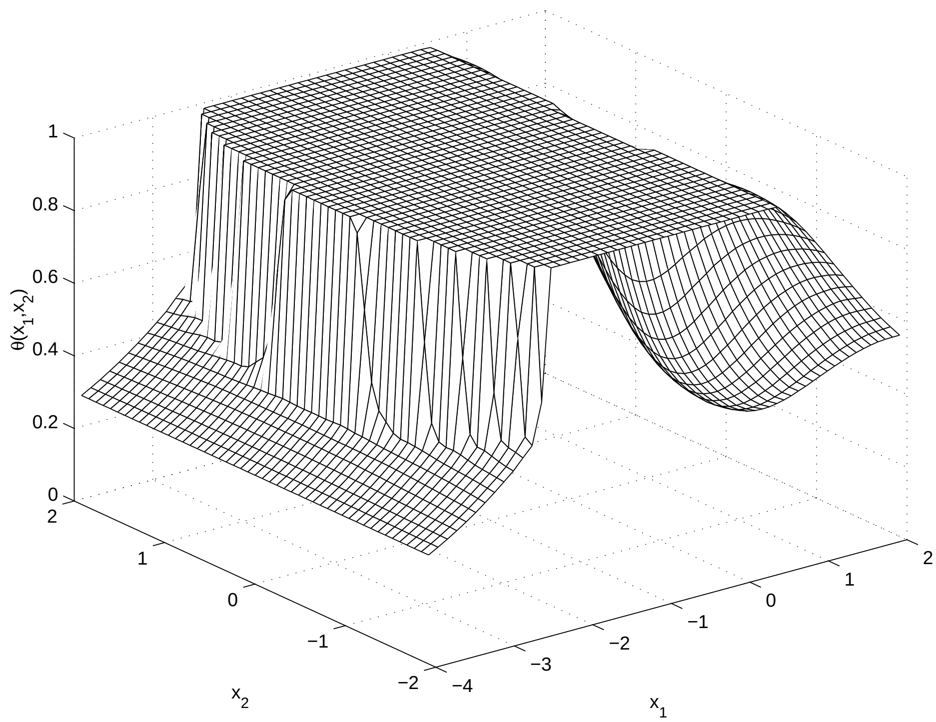


Figure C.12. Homogenized EHL saturation in the whole domain

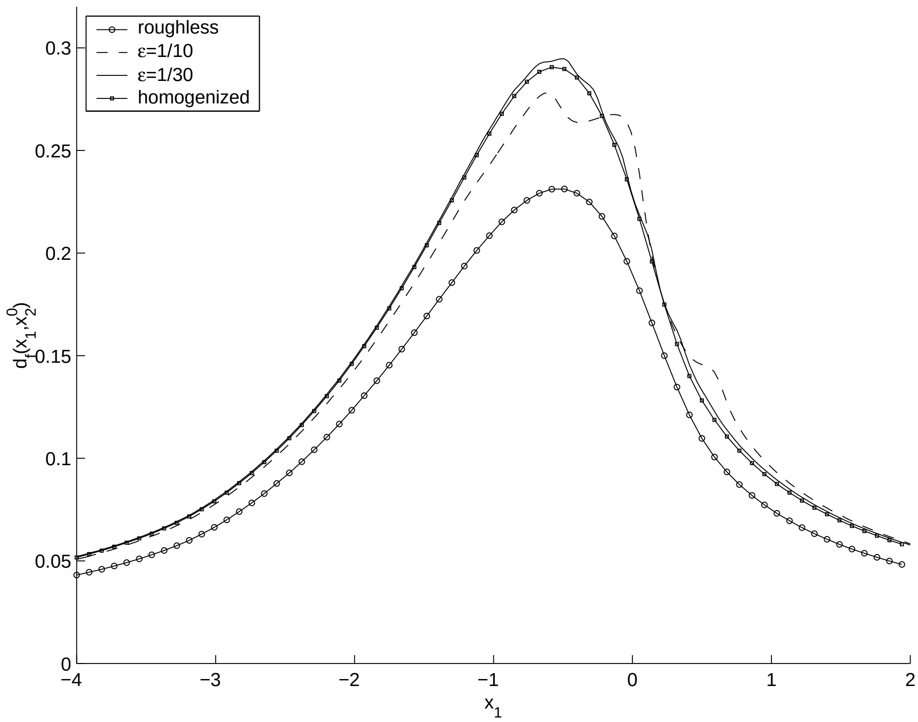


Figure C.13. EHL deformation with transverse roughness patterns at $x_2^0 = 0$

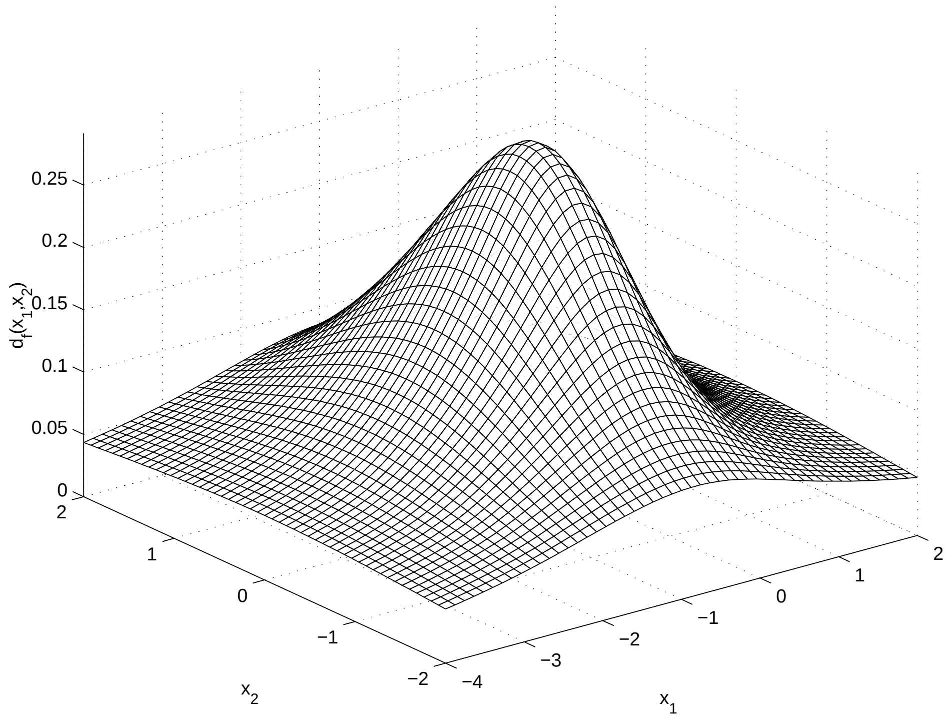


Figure C.14. Homogenized EHL deformation in the whole domain

Longitudinal roughness

Numerical tests have been made for the following rigid contribution to the gap:

$$h_0 + \frac{x_1^2 + x_2^2}{2} + h_0 \ \sin\left(2\pi \frac{x_2 + 2}{4\ \varepsilon}\right)$$

with $h_0 = 0.5$ and different values of ε. Moreover, the piezoviscosity has been taken to $\alpha = 1$. Let us notice that, again, homogenized coefficients in the transverse roughness cases are deduced from TABLE 2.1 (see page 134).

FIG.C.15 and C.16 represent the pressure and deformation profiles at $x_1^0 = -0.4$, in the x_2 direction (in order to observe the roughness effects). This choice corresponds to the maximum pressure in the homogenized case, which is attained at $(x_1^0, x_2^0) = (-0.4, 0)$. Of course, the saturation profile is omitted, for all corresponding saturation functions would be identically equal to 1 (no cavitation in this part of the domain). Significantly, the size of the oscillations for the pressure are damped easily, and convergence of the rough solution to the homogenized one is illustrated on both figures. Similarly, pressure / saturation / deformation curves are omitted, for they are similar to the ones observed in the transverse roughness case.

Influence of the roughness effects in EHL and hydrodynamic cases

FIG.C.17 and C.18 show the difference of roughness effects between a purely hydrodynamic (isoviscous) configuration and an elastohydrodynamic (piezoviscous) configuration. The data are the same as in the transverse roughness case, except for the rough gap whose amplitude of roughness patterns is modified in order to prevent contact between the surfaces in the hydrodynamic case: thus, the gap is

$$h_0 + \frac{x_1^2 + x_2^2}{2} + 0.7\ h_0 \ \sin\left(2\pi \frac{x_1 + 4}{6\ \varepsilon}\right),$$

all other numerical data being the same as before (in particular for the value of h_0 and the piezoviscosity parameter α). It can be noticed that the elevation of the pressure due to the roughness patterns is less important in the EHL case than in the purely hydrodynamic case. This is due to the fact that the elastic deformation tends to damp the additional load corresponding to the roughness. It has little influence on the saturation distribution, although the homogenization process does not allow to get microcavitation effects which do exist when a deterministic rough pattern is considered. Though, this analysis also states that microcavitation effects tend to vanish as ε tends to 0.

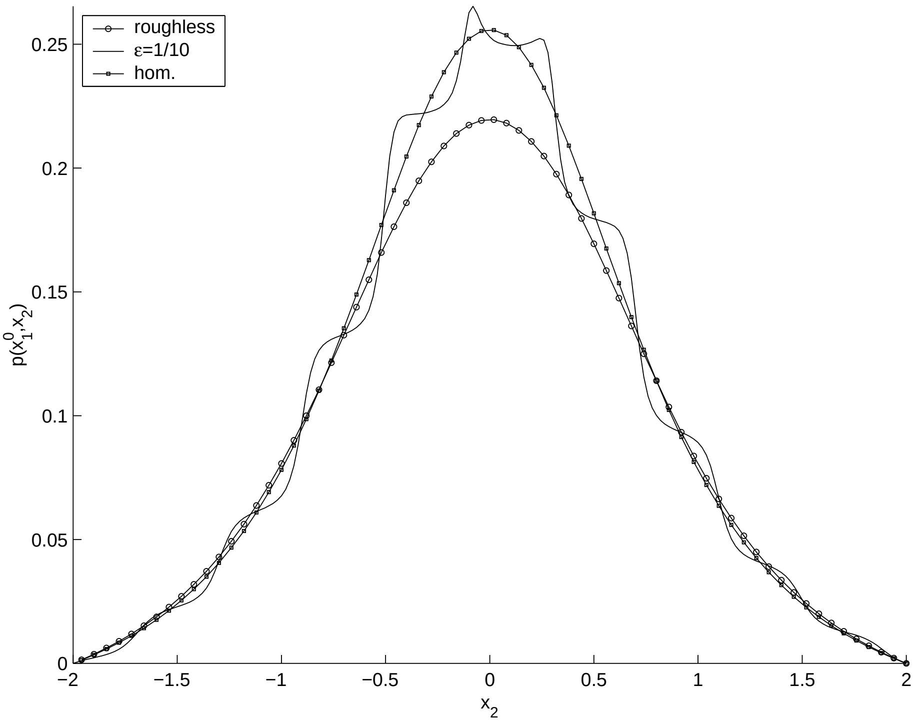


Figure C.15. EHL pressure at $x_1^0 = -0.4$

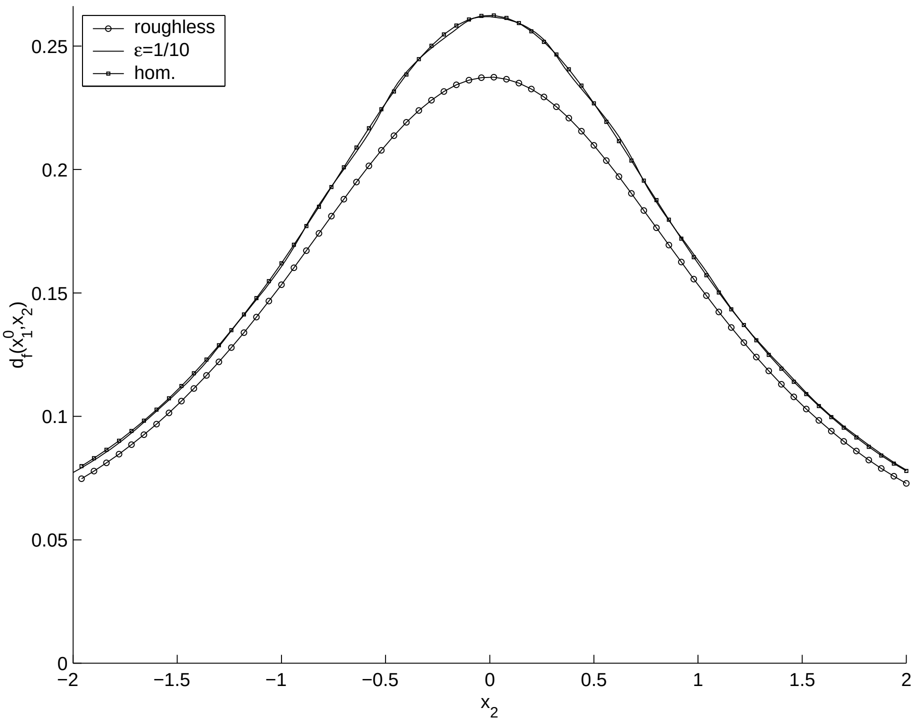


Figure C.16. EHL deformation at $x_1^0 = -0.4$

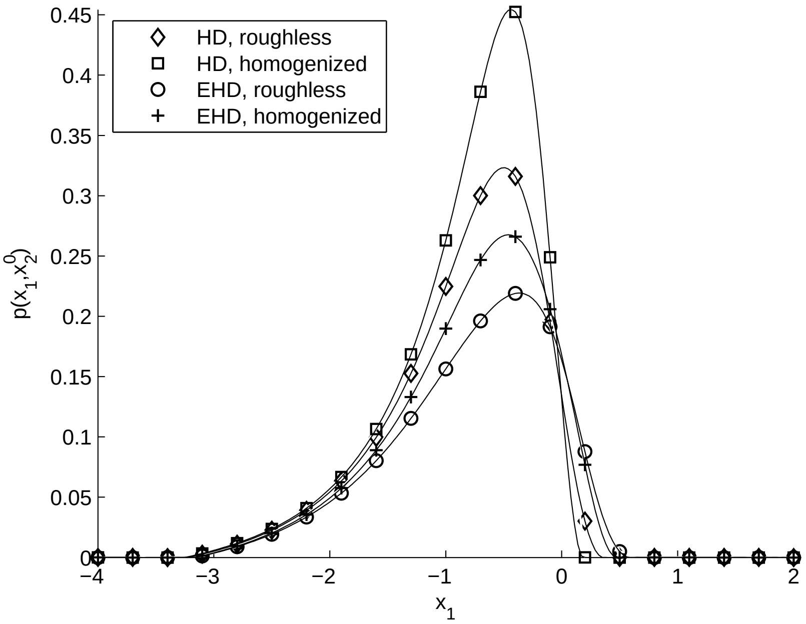


Figure C.17. Transverse roughness effects on the pressure in purely hydrodynamic and elastohydrodynamic cases at $x_2^0 = 0$

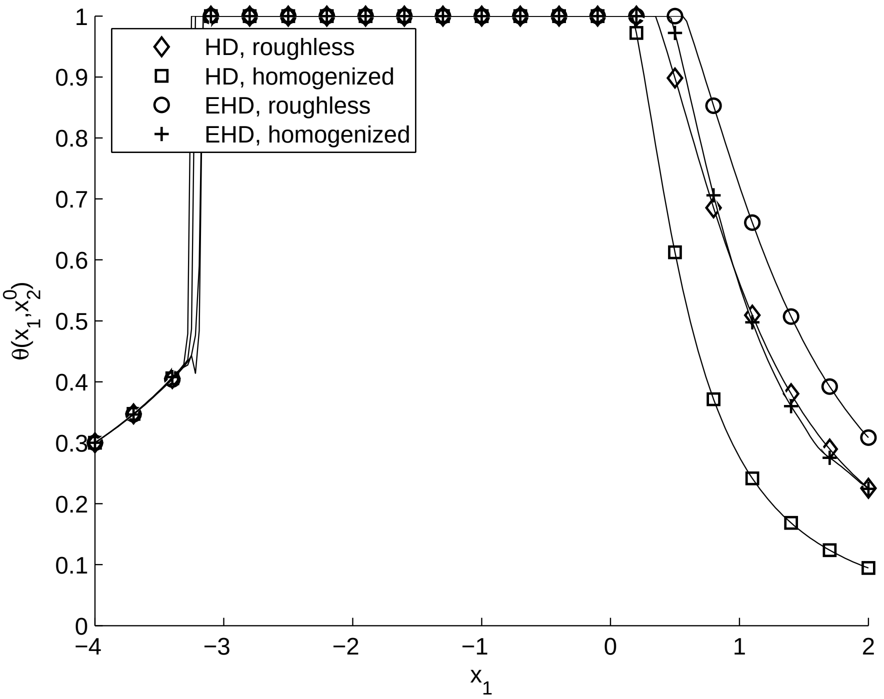


Figure C.18. Transverse roughness effects on the saturation in purely hydrodynamic and elastohydrodynamic cases at $x_2^0 = 0$

Influence of the roughness on the load

Numerical tests have been made for the following rigid contribution to the gap:

$$h_0 + \frac{x_1^2 + x_2^2}{2} + a_r \ \sin\left(2\pi \frac{x_1 + 4}{6\ \varepsilon}\right)$$

with $h_0 \in \{0.5, 1, 1.5, 2\}$ and $a_r/h_0 \in \{0.2, 0.4, 0.6, 0.8, 1\}$. Moreover, the elastic contribution to the gap is the one given at the beginning of Subsection C.3.2 and the piezoviscosity has been taken to $\alpha = 0$ (isoviscous case). Results are given on FIG.C.19, showing the influence of the minimum thickness $h_r - a_r$ on the load W for different values of h_0. Results are taken from the analysis of the corresponding homogenized solution.

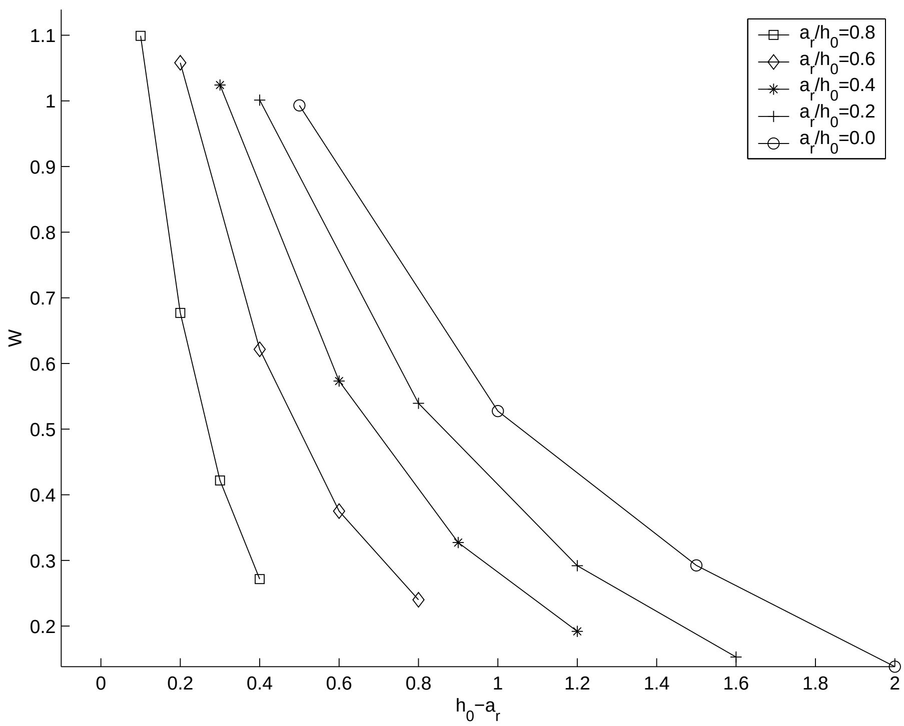


Figure C.19. Influence of the roughness on the load

Influence of the piezoviscosity

We focus on the behaviour of the solution with respect to the piezoviscosity parameter α. Numerical data are the same than in the elastohydrodynamic case with transverse roughness, except that we take into account piezoviscous properties of the lubricant: $\alpha = 0$, 1, 2 or 3.

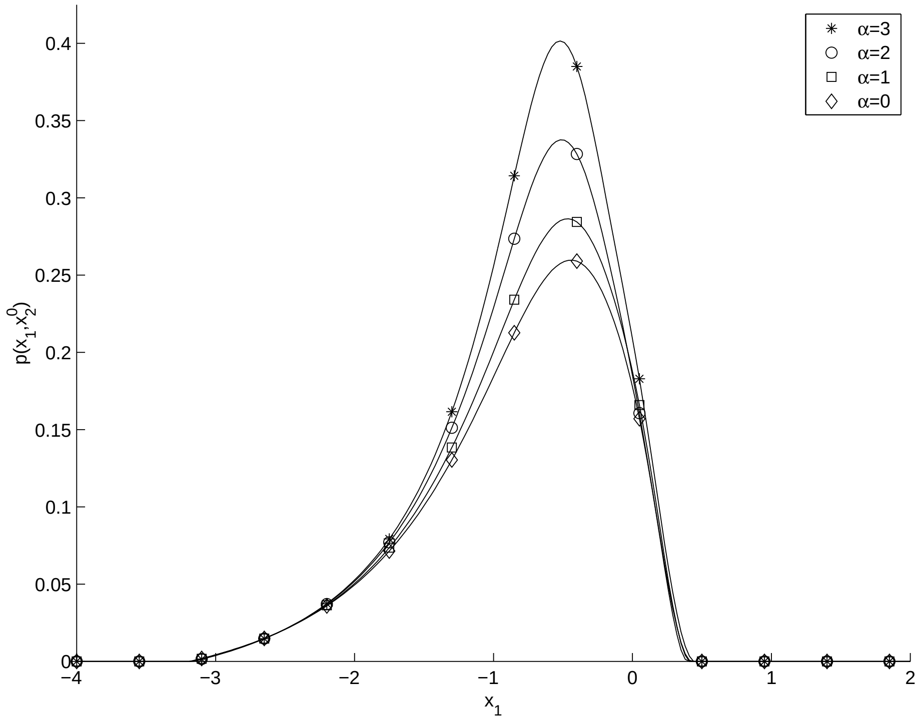


Figure C.20. Influence of the piezoviscosity on the (homogenized) pressure

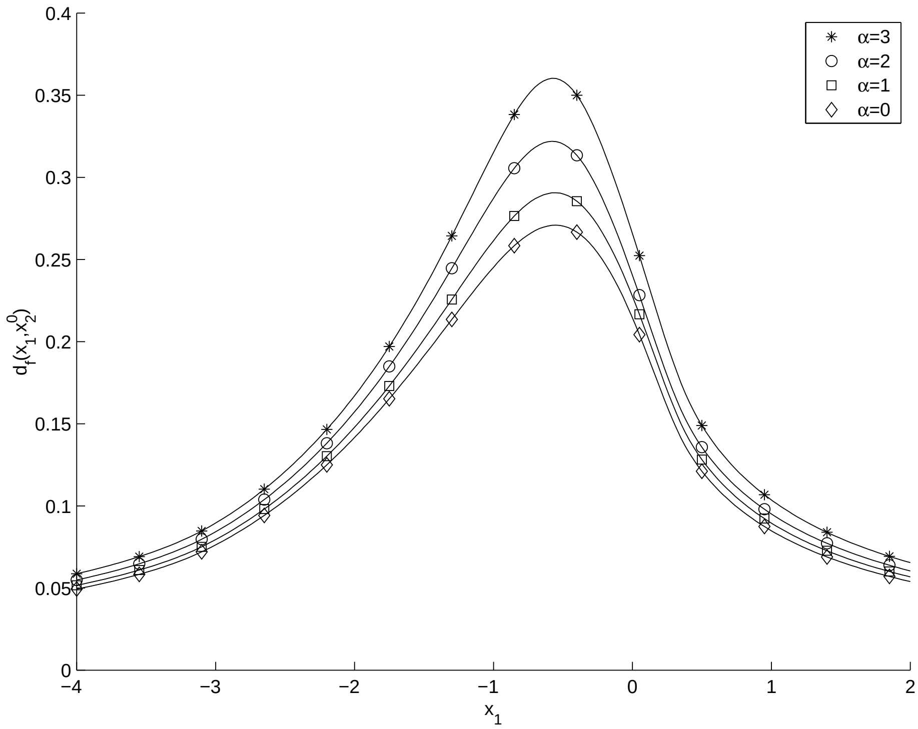


Figure C.21. Influence of the piezoviscosity on the (homogenized) deformation

FIG.C.20 and C.21 represent the pressure and deformation profiles at $x_2^0 = 0$ in the homogenized case. They illustrate the trend induced by the piezoviscosity parameter: the peak pressure and the peak deformation increase with α. Only a few variations affect the saturation distribution (for this reason, the corresponding curves are omitted).

TABLE C.1 (resp. TABLE C.2) gives the variation of the peak pressure (resp. deformation) with respect to the isoviscous case ($\alpha = 0$) in different rough cases (including roughless and homogenized ones).

The relative variation of the peak pressure is denoted

$$\Delta p/p = \frac{\max(p^\alpha) - \max(p^0)}{\max(p^0)}$$

where p^α (resp. p^0) denotes the pressure distribution corresponding to the piezoviscous regime $\alpha \neq 0$ (resp. isoviscous regime $\alpha = 0$). Similarly, the relative variation of the peak deformation is denoted

$$\Delta d/d = \frac{\max(d_f^\alpha) - \max(d_f^0)}{\max(d_f^0)}$$

where d_f^α (resp. d_f^0) denotes the deformation distribution corresponding to the piezoviscous regime $\alpha \neq 0$ (resp. isoviscous regime $\alpha = 0$).

$\Delta p/p$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
roughless	0.0817	0.2633	0.4449
$\varepsilon = 1/10$	0.1035	1.8646	2.8203
$\varepsilon = 1/20$	0.2447	0.6551	1.0192
$\varepsilon = 1/30$	0.1855	0.5161	1.3783
homogenized	0.1041	0.3030	0.5543

Table C.1. Maximum pressure elevation due to piezoviscosity

$\Delta d/d$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
roughless	0.0540	0.1716	0.2787
$\varepsilon = 1/10$	0.0584	0.5742	0.6034
$\varepsilon = 1/20$	0.1078	0.2497	0.4532
$\varepsilon = 1/30$	0.0907	0.2198	0.4386
homogenized	0.0732	0.1895	0.3320

Table C.2. Maximum deformation elevation due to piezoviscosity

C.4 Conclusion

The solution procedure for deterministic periodic roughness computation, which has been developped in [BMV05a], has been extended to the (piezoviscous) elastohydrodynamic case. It is valid for transverse or longitudinal roughness patterns. Further investigation has to be made in order to take into account anisotropic two dimensional effects.

Bibliography

[ACV02] I. Arregui, J. J. Cendán, and C. Vázquez. Mathematical analysis and numerical simulation of a Reynolds-Koiter model for the elastohydrodynamic journal-bearing device. *M2AN Math. Model. Numer. Anal.*, 36(2):325–343, 2002.

[BCV98] G. Bayada, M. Chambat, and C. Vázquez. Characteristics method for the formulation and computation of a free boundary cavitation problem. *J. Comput. Appl. Math.*, 98(2):191–212, 1998.

[BD87] A. Bermúdez and J. Durany. La méthode des caractéristiques pour les problèmes de convection-diffusion stationnaires. *RAIRO Modél. Math. Anal. Numér.*, 21(1):7–26, 1987.

[BF89] G. Bayada and J.-B Faure. A double-scale analysis approach of the Reynolds roughness. Comments and application to the journal bearing. *ASME J. Tribol.*, 111:323–330, 1989.

[BMV05a] G. Bayada, S. Martin, and C. Vázquez. An average flow model of the Reynolds roughness including a mass-flow preserving cavitation model. *ASME J. Tribol.*, 127(4):793–802, 2005.

[BMV05b] G. Bayada, S. Martin, and C. Vázquez. Effets d'anisotropie par homogénéisation dans un problème à frontière libre. *C. R. Math. Acad. Sci. Paris*, 340(7):541–546, 2005.

[DGV02] J. Durany, G. García, and C. Vázquez. Numerical simulation of a lubricated Hertzian contact problem under imposed load. *Finite Elem. Anal. Des.*, 38(7):645–658, 2002.

[DH77] D. Dowson and G. R. Higginson. *Elastohydrodynamic lubrication*. Pergamon Press, Oxford, Great Britain, 1977.

[KB91] A. Kumar and J. F. Booker. A finite element cavitation algorithm. *ASME J. Tribol.*, 113 (2):276–86, 1991.

[SS00] F. Shi and R. F. Salant. A mixed soft elastohydrodynamic lubrication model with interasperity cavitation and surface shear deformation. *ASME J. Tribol.*, 122:308–316, 2000.

[VK90] D. Vijayaraghavan and T. G. Keith. An efficient, robust, and time accurate numerical scheme applied to a cavitation algorithm. *ASME J. Tribol.*, 112:44–51, 1990.